Are the liquidity and collateral roles of asset bubbles different?

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Abstract

Several recent papers introduce different mechanisms to explain why asset bubbles are observed in periods of larger growth. Common assumptions in these papers are the existence of heterogeneous traders and the introduction of borrowing constraints, but they differ in the role of the bubble, that can be used as collateral in a borrowing constraint or to provide liquidities. In this paper, we introduce heterogeneous traders by considering an overlapping generations model with households living three periods. Young households cannot invest in capital, while adults have access to investment and face a borrowing constraint. Introducing bubbles in a quite general way, encompassing the different roles they can have in the existing literature, we show that the bubble may only enhance growth when the borrowing constraint is binding. More significantly, our results do not depend on the role attributed to the bubble.

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1 Introduction

There is a renewed interest to study the interplay between the financial and real spheres of the economy. In particular, several contributions try to understand why episodes of speculative bubbles are associated to periods of economic expansions and why bubble crashes are sources of recession. These phenomena are illustrated in several works, for example, in Caballero *et al.* (2006), Martin and Ventura (2012), Brunnermeier *et al.* (2013) and Kindleberger and Aliber (2015). They were also challenging because seminal papers showed that the existence of rational bubbles in dynamic general equilibrium models was associated to lower GDP per capita (Tirole (1985)) or growth (Grossman and Yanagawa (1993)). This is the so-called crowding-out effect of the bubble.

Most of the papers, that try to reconcile the existence of rational bubbles with the empirical facts, introduce financial imperfections embodied by some borrowing constraints (see Miao (2014) for a short survey) and heterogeneous agents to have different types of traders on the asset markets. T he recent and growing literature about rational bubbles with financial frictions distinguishes between two growth enhancing roles of the bubbles or crowding-in effects. One is the liquidity role of the bubble: agents hold at the beginning of the period the bubble and sell it to increase their productive investment (Cabellero and Krishnamurthy (2006), Kocherlakota (2009), Farhi and Tirole (2012), Martin and Ventura (2012), Hirano and Yanagawa (2017), Kiyotaki and Moore (2018)).[1] The other one is the collaretal role of the bubble: agents buy the bubble to increase their possibilities to borrow and use these loans to invest in capital (Kocherlakota (2009), Miao *et al.* (2015), Martin and Ventura (2016), Bengui and Phan (2018)).

There are different ways to introduce heterogeneous traders. For instance, Kocherlakota (2009), Martin and Ventura (2012), Hirano and Yanagawa (2017), Kiyotaki and Moore (2018) consider heterogeneous investment projects among agents born at the same period, *i.e.* unproductive agents vs. productive ones. Another possibility is to consider overlapping generations with agents living three periods, as in several recent papers like Arce and Lopez-Salido (2011), Farhi and Tirole (2012), Basco (2014, 2016) or Raurich and Seegmuller (2019), among others. These contributions also consider heterogeneous investment projects, but among agents born at different periods. Heterogeneity among agents in terms of investment projects makes room for an asset market that channels liquidities from unproductive agents (lenders-savers) to productive ones (borrowers-investors), which is essential for the existence of the crowding-in effect of the bubble.

In this paper, we consider a three-period lived agents model. We distinguish among three types of traders (young, adult and old), while only two of them may

[1] Note that this liquidity role has also been emphasized in a different perspective by Woodford (1990). Instead of being concerned with bubbles, he focuses on non-neutrality of public debt. In their paper, Kiyotaki and Moore (2018) are interesting to the liquity role of fiat money, which can be seen as a rational bubble. Fiat money allows unproductive entrepreuneurs to transfer some liquidities towards productive one who have an investment opportunity.

buy assets and invest in capital (young and adult). To introduce heterogeneous investments and traders, we assume that young households do not hold capital, while adults invest in this asset expecting a positive return. It means that adults are the most and only productive investors. At each period of time, there is also a credit market in which young households and adults can make deposits and borrow. The amount of credit is limited by a borrowing constraint.

We start by introducing bubbles considering two examples where, at the second period of life, borrowing is constrained and collateralized by capital, *i.e.* a fundamental collateral. In the first one, the bubble is bought by young households and sold when adult. Therefore, the adult can sell the bubble to invest more in capital. Selling the bubble corresponds to a transfer from the unproductive young agents to productive adults. This mechanism extends to a general equilibrium framework the liquidity effect of the bubble developed in Farhi and Tirole (2012), which is also in line with many other existing papers like Hirano and Yanagawa (2017), Kocherlakota (2009) or Martin and Ventura (2012). In the second example, following Kocherlakota (2009) or Martin and Ventura (2016), the bubble is only bought by adults and is used as a collateral in the borrowing constraint. By increasing the collateral, the bubble increases the amount borrowed, promoting a higher investment in capital. Therefore, these two examples illustrate the two different roles of bubbles in the economy. We show that these two approaches lead to exactly the same equilibrium, despite the fact that the two mechanisms of the bubble seem to be a priori different.

Then, we propose a general model that encompasses the two previous examples and in which bubbles may have both the liquidity and collateral roles. To this end, we assume that both young and adult households may buy or sell short the bubble. Of course, the bubble bought by adults still plays the role of a bubbly collateral and capital plays the role of a fundamental collateral in the borrowing constraint. To fix ideas and to be able to analyze the dynamics in a simple way, firms produce the final good using an Ak technology, which implies endogenous growth.

We first analyze the model without a borrowing constraint. Despite the fact that there are heterogeneous traders, the bubble has a crowding-out effect on growth. As in the seminal contribution of Grossman and Yanagawa (1993), the bubbly BGP is always characterized by a lower growth than the bubbleless one.

When the borrowing constraint is binding, the existence of an investment multiplier promotes the positive effect of the bubble on capital, whereas the resulting increase of the interest rate has a negative effect. When the degree of pledgeability of the fundamental collateral is small enough, the first effect dominates and the bubble enhances growth. On the contrary, when the degree of pledgeability is sufficiently large, the bubble has a crowding-out effect on growth. The main conclusion of our paper is that these results do not depend on the particular type of bubble considered. This means that there is no distinction between the liquidity and the collateral roles of the bubble.

In the following section, we present the two examples of models where the bubble is either bought when young and used to provide liquidities or used as a collateral when adult. In Section 3, we introduce our general model of bubbles, considering first the framework without binding borrowing constraint. A binding borrowing constraint is introduced in a second step to show its role on the crowding-in effect of the bubble whatever the type of bubble considered. Section 4 concludes and technical details are relegated to the Appendix.

2 Two examples of models with heterogeneous agents, borrowing constraints and bubbles

To motivate our general approach of bubbles, we start by presenting two models with heterogeneous traders and borrowing constraints. Such a heterogeneity of investment projects can be introduced in models with infinitely-lived agents (Kocherlakota (2009), Hirano and Yanagawa (2017)) or two-period lived agents (Martin and Ventura (2012, 2016)). As in several recent papers (Arce and Lopez-Salido (2011), Basco (2014, 2016), Farhi and Tirole (2012), Raurich and Seegmuller (2019)), we introduce heterogeneity among agents in terms of investment projects considering an overlapping generations model with three-period lived agents. Therefore, agents may invest both when young or adult. Nevertheless, in accordance with Farhi and Tirole (2012), young households do not invest in capital, while adults expect a positive return for their capital investment.[2] It means that adults are the most and only productive investors. This framework allows to consider the two roles of the bubble. In the first model we present, the bubble has a liquidity role when an adult household sells the bubble bought when young to invest in capital. In the second model, the bubble is bought by adult agents and it plays the role of a collateral in the borrowing constraint.[3]

2.1 Model with bubble bought by young savers, YS

The first model illustrates the liquidity role of the bubble. Agents buy the bubble when they are savers and, when they become investors, they sell it to increase investment in capital. This liquidity role was introduced in Farhi and Tirole (2012). In this example, that we denote as YS because young agents are savers, we introduce it in a general equilibrium framework. The mechanism it describes is also in line with many other existing papers like Hirano and Yanagawa (2017), Kocherlakota (2009), or Martin and Ventura (2012).

We consider an overlapping generations economy populated by agents living for three periods. An agent is young in the first period of life, adult in the second period and old in the third period. There is no population growth. The population size of a generation is constant and normalized to one.

Each household obtains utility from consumption at each period of time. Preferences of an individual born in period t are represented by the following

[2] Note that in a recent paper, Raurich and Seegmuller (2019) already investigate what happens when the young invest in capital, while adults have not access to the capital market.

[3] In this paper, agents which trade the different assets are identified as households, whereas some papers rather speak about entrepreneurs (Kocherlakota (2009), Farhi and Tirole (2012), Hirano and Yanagawa (2017)). The difference only concerns the denomination of the agents.

utility function:

$$\alpha u_1(c_{1,t}) + \beta u_2(c_{2,t+1}) + \gamma u_3(c_{3,t+2}) \tag{1}$$

where $\alpha, \beta, \gamma > 0$ and the utility functions $u_i(c_i)$ are well defined on $\mathbb{R}_+$, strictly increasing ($u_i'(c_i) > 0$) and concave ($u_i''(c_i) < 0$) on $\mathbb{R}_{++}$.

The household supplies one unit of labor when young and adult. When young, labor efficiency is one, while it is equal to $\phi > 0$ when adult.

There are three assets in the economy: capital used in the production k_t, deposits that allows to finance loans d_{it}, and an asset without fundamental value supplied in one unit, with a price b_{1t}. There is a bubble as soon as $b_{1t} > 0$.

When young, the household invests in deposits d_{1t} and can buy the bubble b_{1t}. In the next period, these two assets provide returns given by R^d_{t+1} and R_{t+1}, respectively. When adult, the household can only invest in deposit d_{2t+1} and capital k_{t+2}. When old, these two assets are remunerated with the returns R^d_{t+2} and q_{t+2}, respectively. Of course, when $d_{it} < 0$, the household contracts loans. When adult, these loans are limited by the following borrowing constraint:

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2} \tag{2}$$

where $\theta \in [0, 1)$ is the degree of pledgeability. This constraint means that the household can borrow an amount $d_{t+2} < 0$, as long as the repayment does not exceed a fraction θ of the future return from her productive investment at period $t + 2$. The parameter θ is related to the financial market imperfection, with a lower θ meaning a stronger imperfection. Young agents will not face an equivalent constraint, because they will not be short-sellers of the liquid assets, deposits and the bubble, but rather use them to make transfers to the next periods of life.

The budget constraints when young, adult and old faced by a household born in period t are, respectively:

$$c_{1t} + b_{1t} + d_{1t} = w_t \tag{3}$$

$$c_{2t+1} + k_{t+2} + d_{2t+1} = \phi w_{t+1} + R_{t+1} b_{1t} + R^d_{t+1} d_{1t} \tag{4}$$

$$c_{3t+2} = q_{t+2} k_{t+2} + R^d_{t+2} d_{2t+1} \tag{5}$$

The household maximizes the utility (1) under the constraints (2)-(5). We focus on equilibria where the borrowing constraint is binding. Using the first order conditions, we deduce that $R_{t+1} = R^d_{t+1}$ and:

$$u_1'(c_{1t}) = \frac{\beta}{\alpha} R_{t+1} u_2'(c_{2,t+1}) \tag{6}$$

$$u_2'(c_{2,t+1}) = \frac{\gamma}{\beta} u_3'(c_{3t+2}) \frac{q_{t+2}(1-\theta)}{1 - \theta q_{t+2}/R_{t+2}} \tag{7}$$

In addition, the borrowing constraint is binding if:

$$u_2'(c_{2,t+1}) > R_{t+2} \frac{\gamma}{\beta} u_3'(c_{3,t+2}) \tag{8}$$

Using (7), this is ensured by $q_{t+2} > R_{t+2} > \theta q_{t+2}$. Note that the equilibrium cannot satisfy $R_{t+2} > q_{t+2}$ because, in this case, adults will not invest in capital since d_{2t+1} gives a higher return. Note also that $R_{t+2} < \theta q_{t+2}$ cannot occur as it would imply that an adult can borrow an infinite amount to invest in capital without being constrained.

Finally, the bubble evolves according to:

$$b_{1t+1} = R_{t+1} b_{1t} \tag{9}$$

where R_{t+1} also measures the growth of the asset price bubble b_{1t+1}/b_{1t}.

Using the binding borrowing constraint and the market clearing on deposits $d_{1t} + d_{2t} = 0$, we have $-d_{2t} = d_{1t} = \frac{\theta q_{t+1} k_{t+1}}{R_{t+1}}$. Then, using the budget constraints (3)-(5), the consumptions are given by:

$$\begin{aligned} c_{1t} &= w_t - b_{1t} - \frac{\theta q_{t+1} k_{t+1}}{R_{t+1}} && (10) \\ c_{2t+1} &= \phi w_{t+1} + b_{1t+1} + \theta q_{t+1} k_{t+1} - k_{t+2}\left(1 - \frac{\theta q_{t+2}}{R_{t+2}}\right) && (11) \\ c_{3t+2} &= (1-\theta) q_{t+2} k_{t+2} && (12) \end{aligned}$$

We will obtain an equilibrium of this economy substituting these three equations in the two arbitrage conditions (6) and (7).

2.2 Model with bubble bought by adult investors, AI

This model, that we called AI because adults are investors, illustrates the collateral role of the bubble introduced by Kocherlakota (2009) and Martin and Ventura (2016). Investors buy the bubble to use it as collateral of credits that finance investment in capital.

The model is the same as the YS, except for the bubble. There is still an asset without fundamental value supplied in one unit, but it is not bought by young households and sold by adults. Instead, it is bought by adults at the price b_{2t+1} and sold by old households. Therefore, the budget constraints become:

$$c_{1t} + d_{1t} = w_t \tag{13}$$

$$c_{2t+1} + k_{t+2} + d_{2t+1} + b_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t} \tag{14}$$

$$c_{3t+2} = q_{t+2} k_{t+2} + R^d_{t+2} d_{2t+1} + R_{t+2} b_{2t+1} \tag{15}$$

Since the household may buy the bubble when she also invests in capital, we follow Martin and Ventura (2016) assuming that the bubble is also used as collateral to borrow. We have now both fundamental and bubbly collaterals. The borrowing constraint writes now:

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2} + R_{t+2} b_{2t+1} \tag{16}$$

The household can borrow an amount $d_{t+2} < 0$, as long as the repayment does not exceed θ of the future return from her productive investment and the market value of the bubble at period $t+2$. Kocherlakota (2009) also introduces a constraint where the bubble has the role of collateral. It corresponds to the case where $\theta = 0$ (see also Section 3.6.2).

Solving the household problem, we find $R^d_{t+2} = R_{t+2}$. When the borrowing constraint is binding, we also get the first order conditions (6)-(8). They imply again that the borrowing constraint is binding when $q_{t+2} > R_{t+2} > \theta q_{t+2}$.

At an equilibrium, the binding borrowing constraint and the market clearing condition on deposits, $d_{1t} + d_{2t} = 0$, imply that:

$$-d_{2t} = d_{1t} = \frac{\theta q_{t+1} k_{t+1}}{R_{t+1}} + b_{2t} \tag{17}$$

Moreover, the bubble evolves according to:

$$b_{2t+1} = R_{t+1} b_{2t} \tag{18}$$

Using (17) and (18) and the budget constraints (13)-(15), we derive the same expressions for c_{1t}, c_{2t+1} and c_{3t+2} than (10)-(12), replacing b_{1t} by b_{2t}. Therefore, once we substitute b_{1t} by b_{2t}, the equilibrium is defined exactly by the same equations than in the previous model.

2.3 In Short

The equilibrium of the models YS and AI is characterized by the same equations, which means that they share the same equilibrium. There is a perfect equivalence between the models YS and AI, despite the fact that the role of the bubble is a priori different. In the model YS, the bubble is introduced to provide liquidity to the investors, *i.e.* adult households. In contrast, in the model AI, the bubble is bought by adult/investors, using it as a collateral to borrow. Since the reduced forms of the two models are identical, these two mechanisms play exactly the same role and the bubble has finally the same effect in both models.

3 A general approach with bubble bought by young savers and adult investors

To confirm this equivalence result, we develop now a more general model that encompasses both formulations - it will admit the models YS and AI as particular cases - and allows us to study the role of bubble in a more general way. In the following, we start by presenting the production. Then, we will present our general framework, but without any binding borrowing constraint, to show that even if there are heterogeneous traders, bubbles cannot enhance growth. Finally, we introduce a borrowing constraint and discuss in details the crowding-in versus crowding-out effects of the bubble.

3.1 Production sector

To simplify our dynamic analysis, we introduce a simple Ak type technology. Aggregate output is produced by a continuum of firms, of unit size, using labor, l_t, and capital, k_t, as inputs. In addition, production benefits from an externality that summarizes a learning by doing process and allows to have sustained growth. Following Frankel (1962) or Ljungqvist and Sargent (2004, chapter 14), this externality depends on the average capital-labor ratio.

Letting $a_t \equiv k_t/l_t$, $\overline{a}_t$ represents the average ratio of capital over labor. Firms produce the final good using the following technology:

$$y_t = F(k_t, \bar{a}_t l_t)$$

The technology $F(k_t, \bar{a}_t l_t)$ has the usual neoclassical properties, *i.e.* it is a strictly increasing and concave production function satisfying the Inada conditions, and is homogeneous of degree one with respect to its two arguments.

Profit maximization under perfect competition implies that the wage w_t and the return of capital q_t are given by:[4]

$$w_t = F_2(k_t, \bar{a}_t l_t)\bar{a}_t \tag{19}$$

$$q_t = F_1(k_t, \bar{a}_t l_t) \tag{20}$$

All equilibria we will consider are symmetric ones, *i.e.* $a_t = \overline{a}_t$. Let us define $s \equiv F_1(1,1)/F(1,1) \in (0,1)$ the capital share in total production and $A \equiv F(1,1) > 0$. Using (19) and (20), we deduce that:

$$w_t = (1-s)Aa_t \equiv w(a_t) \tag{21}$$

$$q_t = sA \tag{22}$$

which give the wage and the return of capital at an equilibrium.

3.2 The model without binding borrowing constraint

We generalize the models of Section 2 by providing a framework that encompasses them. To disentangle between the roles of heterogeneous traders and borrowing constraints, we first consider a model with a perfect credit market.

The main change with respect to the YS and AI models concerns the asset without fundamental value which is a bubble if it is positively valued. We introduce bubbles in a more general way, which encompasses these two models. There are two possible interpretations of our framework. One possibility is to assume that there are two assets without fundamental value that are bubbles if their prices are positive. In this case, a household can buy one of these two assets when young at price b_{1t} and the other one when adult at price b_{2t+1}. They have *a priori* different returns, denoted by r_{t+1} and R_{t+2}, respectively. Another possibility is to assume that there is only one asset without fundamental value supplied in one unit. Then, b_{1t} and b_{2t+1} represent the price times the share of

[4] We denote by $F_i(.,.)$ the derivative with respect to the ith argument of the function.

this asset bought when young and adult, respectively. Of course, in such a case, the returns of b_{1t} and b_{2t} are the same, *i.e.* $r_t = R_t$ for all t. Moreover, having $b_{it} < 0$ for $i = 1$ or $i = 2$ becomes possible. This means that the household is a short-seller of this asset at time t.

Accordingly, the budget constraints write now:

$$c_{1t} + d_{1t} + b_{1t} = w_t \tag{23}$$

$$c_{2t+1} + k_{t+2} + d_{2t+1} + b_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t} + r_{t+1} b_{1t} \tag{24}$$

$$c_{3t+2} = q_{t+2} k_{t+2} + R^d_{t+2} d_{2t+1} + R_{t+2} b_{2t+1} \tag{25}$$

A household maximizes the utiliy (1) under the budget constraints (23)-(25). Solving the household problem, we deduce that all assets are substitutable, *i.e.* $q_{t+1} = R^d_{t+1} = r_{t+1} = R_{t+1}$, and therefore:

$$u_1'(c_{1t}) = \frac{\beta}{\alpha} R_{t+1} u_2'(c_{2t+1}) \tag{26}$$

$$u_2'(c_{2t+1}) = \frac{\gamma}{\beta} R_{t+2} u_3'(c_{3t+2}) \tag{27}$$

Let us introduce $x_{1t} = b_{1t} + d_{1t}$ and $x_{2t} = b_{2t} + d_{2t}$. This means that $x_{1t} + x_{2t} = b_{1t} + b_{2t}$ because $d_{1t} + d_{2t} = 0$ at the equilibrium on the deposit market. Therefore, we distinguish between two situations:

- $x_{1t} + x_{2t} = 0$ if there is no bubble, *i.e.* $b_{1t} = b_{2t} = 0$;
- $x_{1t} + x_{2t} > 0$ if there is a bubble on at least one asset, *i.e.* $b_{1t} > 0$ and/or $b_{2t} > 0$, or if b_{1t} and b_{2t} represent the same asset, in which case a bubble exists if $b_{1t} + b_{2t} > 0$, but either b_{1t} or b_{2t} may be negative.

In any case, the assets evolve according to:

$$x_{1t+1} + x_{2t+1} = R_{t+1}(x_{1t} + x_{2t}) \tag{28}$$

Note that this equation encompasses the different form of bubbles introduced in the models YS and AI, and is even more general.

Using (23)-(25) and the equality between the different asset returns, the budget constraint on the life-cycle writes:

$$c_{1t} + \frac{c_{2,t+1}}{R_{t+1}} + \frac{c_{3t+2}}{R_{t+1}R_{t+2}} = w_t + \frac{\phi w_{t+1}}{R_{t+1}} \tag{29}$$

When the utility is log-linear, *i.e.* $u_i(c_i) = \ln c_i$ for $i = 1, 2, 3$, we easily deduce that the consumptions are given by:

$$c_{1t} = \frac{\alpha}{\alpha+\beta+\gamma}\left(w_t + \frac{\phi w_{t+1}}{R_{t+1}}\right) \tag{30}$$

$$c_{2t+1} = \frac{\beta R_{t+1}}{\alpha+\beta+\gamma}\left(w_t + \frac{\phi w_{t+1}}{R_{t+1}}\right) \tag{31}$$

$$c_{3t+2} = \frac{\gamma R_{t+1} R_{t+2}}{\alpha+\beta+\gamma}\left(w_t + \frac{\phi w_{t+1}}{R_{t+1}}\right) \tag{32}$$

Using the budget constraints (23)-(25), $x_{1t} = b_{1t} + d_{1t}$ and $x_{2t} = b_{2t} + d_{2t}$, we obtain:

$$x_{1t} = \frac{\beta+\gamma}{\alpha+\beta+\gamma} w_t - \frac{\alpha\phi w_{t+1}/R_{t+1}}{\alpha+\beta+\gamma} \tag{33}$$

$$k_{t+2} + x_{2t+1} = \frac{\gamma}{\alpha+\beta+\gamma}(\phi w_{t+1} + R_{t+1} w_t) \tag{34}$$

Taking into account that the population size of each generation is constant and normalized to one, the equilibrium in the labor market in efficient units requires $l_t = 1 + \phi$. Hence, we have $k_{t+1} = a_{t+1} l_{t+1} = a_{t+1}(1+\phi)$.

Let us define $\widetilde{x}_{it} \equiv x_{it}/[(1+\phi)a_t]$ and $g_{t+1} \equiv a_{t+1}/a_t$. Using (21) and (22), equations (28), (33) and (34) rewrite:

$$b_{t+1} g_{t+1} = sAb_t \tag{35}$$

$$\widetilde{x}_{1t} = \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)} g_{t+1} \tag{36}$$

$$\widetilde{x}_{2t} = \frac{\gamma}{\alpha+\beta+\gamma}\left(\frac{\phi(1-s)A}{1+\phi} + \frac{s(1-s)A^2}{1+\phi}\frac{1}{g_t}\right) - g_{t+1} \tag{37}$$

with:

$$\begin{aligned} b_t \equiv \widetilde{x}_{1t} + \widetilde{x}_{2t} &= \frac{\beta+\gamma(1+\phi)}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} + \frac{\gamma}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{1+\phi}\frac{1}{g_t} \\ &\quad - g_{t+1}\left[1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}\right] \equiv \Omega(g_t, g_{t+1}) \end{aligned} \tag{38}$$

By inspection of equation (35), two BGPs may exist, a bubbly one and a bubbleless one.

Proposition 1 *When there is no binding borrowing constraint, a bubbly BGP (b^*, g^*) coexsists with the bubbleless BGP $(0, g^{**})$ if:*

$$\frac{\alpha(s+\phi) + (\beta+\gamma)s(1+\phi)}{(1-s)[\beta+\gamma(2+\phi)]} < 1 \tag{39}$$

In addition, we always have $g^* < g^{**}$. Finally, the bubbly BGP is a saddle, the bubbleless BGP is a locally indeterminate sink, and one converges to these two steady states with oscillations.

Proof. See Appendix A. ∎

The bubbleless BGP always exists, while the existence of the bubbly one requires (39). Indeed, by direct inspection of equation (38), there is a positive bubble when the growth factor is equal to the interest factor sA if the saving is high enough.

The more interesting result follows from the comparison between the bubbly and the bubbleless BGPs. Proposition 1 shows that growth is lower with a bubble than without. We deduce that without a binding borrowing constraint, the bubble has a crowding-out effect since it reduces growth. We can understand this result by computing total savings at time t. Using (33) and (34), the sum of the savings of young households and adults is equal to:

$$k_{t+1} + x_{1t} + x_{2t} = \frac{\gamma}{\alpha+\beta+\gamma} R_t w_{t-1} + \frac{\beta+\gamma(1+\phi)}{\alpha+\beta+\gamma} w_t - \frac{\alpha\phi w_{t+1}/R_{t+1}}{\alpha+\beta+\gamma} \quad (40)$$

Total savings does not depend on the existence of the bubble (see the right-hand side of equation (40)). This implies that the bubble has a crowding-out effect on future capital. It is important to outline that this result occurs despite the fact that there are heterogeneous traders (young and adult) with heterogeneous opportunities to invest in capital (strictly positive return for an adult and zero return for a young).

3.3 Households face binding borrowing constraints

We introduce in the previous model a borrowing constraint that restricts the amount borrowed by people investing in capital, *i.e.* adults. We adopt the borrowing constraint introduced in the model AI (Section 2.2): the household can borrow an amount $d_{2t+1} < 0$, as long as the repayment does not exceed a fraction θ of the future return from her productive investment and the market value of the bubble at period $t+2$. As in the model AI, two assets serve as a collateral: the capital and the bubble bought at the adult age. The first one can be identified as a fundamental collateral and the second one as a bubbly one. The degree of pledgeability of the fundamental collateral is measured by $\theta \in [0, 1)$. The borrowing constraint is then given by:

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2} + R_{t+2} b_{2t+1} \quad (41)$$

A household maximizes her utility function (1) facing the budget constraints (23)-(25) and the borrowing constraint (41). Solving the household problem, we deduce that $R^d_{t+1} = r_{t+1}$, $R^d_{t+2} = R_{t+2}$ and therefore:

$$u'_1(c_{1t}) = \frac{\beta}{\alpha} R_{t+1} u'_2(c_{2,t+1}) \quad (42)$$

When the borrowing constraint is binding, *i.e.* $u'_2(c_{2,t+1}) > R_{t+2}\frac{\gamma}{\beta}u'_3(c_{3t+2})$, we also have:

$$u'_2(c_{2,t+1}) = \frac{\gamma}{\beta} u'_3(c_{3t+2}) \frac{q_{t+2}(1-\theta)}{1-\theta q_{t+2}/R_{t+2}} \quad (43)$$

which implies that the constraint is binding if:

$$q_{t+2} > R_{t+2} > \theta q_{t+2} \quad (44)$$

Using $x_{1t} = b_{1t} + d_{1t}$ and $x_{2t} = b_{2t} + d_{2t}$, the binding constraint becomes:

$$R_{t+2} x_{2t+1} = -\theta q_{t+2} k_{t+2} \quad (45)$$

and (28),

$$R_{t+1}x_{1t} = x_{1t+1} + x_{2t+1} + \theta q_{t+1}k_{t+1} \tag{46}$$

Substituting (28), (45) and (46) in the budget constraints (23)-(25), we get:

$$\begin{aligned}
c_{1t} &= w_t - x_{1t} - x_{2t} - \theta\frac{q_{t+1}k_{t+1}}{R_{t+1}} && (47)\\
c_{2t+1} &= \phi w_{t+1} + x_{1t+1} + x_{2t+1} + \theta q_{t+1}k_{t+1} - k_{t+2}\left(1 - \frac{\theta q_{t+2}}{R_{t+2}}\right) && (48)\\
c_{3t+2} &= (1-\theta)q_{t+2}k_{t+2} && (49)
\end{aligned}$$

We note that the first order conditions (42) and (43), and the condition to ensure a binding credit constraint (44) are identical to the conditions in the models YS and AI. This holds whatever the utility functions $u_i(c_i)$ are. If we compare the consumptions, we observe that (47)-(49) are a generalization of equations (10)-(12) whether b_{1t} is substituted or not by b_{2t}. Indeed, when $x_{1t}+x_{2t} = b_{1t}$, we obtain the model YS where the bubbly asset is bought when young only, to provide liquidity in the adult age. When $x_{1t}+x_{2t} = b_{2t}$, we obtain the model AI, where adults buy the bubble also to relax the credit constraint. The approach developed in this section encompasses the two previous models, generalizes them, and allows us to consider other configurations. We can for instance think about the situation where both b_{1t} and b_{2t+1} are positive, *i.e.* the bubbly asset is bought both by young households and adults, or when b_{1t} is positive but b_{2t+1} is negative, meaning that adults are short-sellers of the bubble to finance capital investment, while young agents buy this asset which gives it a positive value.

3.4 Discussion of the related literature

Before analyzing the equilibrium in detail and the crowding-in versus crowding-out effect of the bubble, we can more specifically compare our model with the closest related literature, namely Farhi and Tirole (2012), Hirano and Yanagawa (2017), Martin and Ventura (2012, 2016) and Kocherlakota (2009).

Farhi and Tirole (2012) is a special case of the model with $b_{2t} = 0$ for all t. We generalize their approach considering a general equilibrium model, meaning that prices and incomes are endogenous. We also note that we do not have to introduce an adding asset representing outside liquidity, called trees, which is required for their result.

Hirano and Yanagawa (2017) is quite similar to our framework when $b_{2t} = 0$ for all t. Despite the fact that they consider infinitely-lived agents, they distinguish between high and low productive agents. Young and adult households correspond to them in our framework, taking the extreme case where the first ones are completely unproductive. Having $b_{2t} = 0$ in our framework means that the bubble is used only to transfer resources from less to more productive agents, as in their paper.

The mechanism for the crowding-in effect of the bubble highlighted by Martin and Ventura (2012) is also encompassed in our framework. They consider two-period lived overlapping generations, in which agents are heterogeneous because their investments have different returns. Despite the fact that they have no credit, the bubble enhances growth because it is used to reallocate resources from less to more productive traders, as in our framework with $b_{1t} > 0$ and $b_{2t} = 0$. In our model, it corresponds to a situation where the unproductive young agents buy the bubble from the productive adult ones. Note that in contrast to Martin and Ventura (2012), we will not need any exogenous bubble shocks to have a crowding-in effect of the bubble.

Our model also generalizes Martin and Ventura (2016) when $b_{1t} = 0$ for all t. Indeed, we also have workers that provide credit (young agents in our framework) to some investors (adults in our framework), but this heterogeneity among agents comes from the lifetime of three periods. In their model, the credit constraint also has two types of collateral, one related to the value of the firm and one associated to the bubble. However, note that in contrast to Martin and Ventura (2016), bubbles have a crowding-in effect without adding bubble shocks in our paper.

The type of credit constraint investigated by Kocherlakota (2009) can also be seen as a particular case of our model in which we set $\theta = 0$ (see also Section 3.6.2). In Kocherlakota (2009), both productive and unproductive traders hold the bubble, which corresponds to the case where $b_{it} > 0$ for $i = 1, 2$ in our framework. The young savers, which represent the unproductive traders, do not however face any binding borrowing constraint in our model. However, it is important to see that this case where $\theta = 0$ corresponds to a configuration with strong financial imperfections in our framework, since capital does no more play the role of collateral.

3.5 Constrained equilibrium

We characterize the constrained equilibrium. To obtain simple and easily interpretable expressions, we assume in the rest of the paper a log-linear utility function, meaning that $u_i(c_i) = \ln c_i$ for all $i = 1, 2, 3$.

Substituting (21), (22) and (47)-(49) in (43) evaluated one period before, we obtain:

$$a_{t+1} = \frac{\gamma}{(\beta+\gamma)(1+\phi)} \frac{\phi w(a_t) + x_{1t} + x_{2t} + \theta q_t (1+\phi) a_t}{1 - \theta q_{t+1}/R_{t+1}} \equiv a^s_{t+1} \tag{50}$$

This equation determines the amount invested in capital given the assets hold from the previous period. We call it the asset supply, a^s_{t+1}, because despite her labor income, an adult sell the bubble $x_{1t} + x_{2t}$ and the deposits corresponding to the fundamental collateral $\theta q_t(1+\phi)a_t$ to finance her investment in capital. The term $1/(1 - \theta q_{t+1}/R_{t+1})$ corresponds to an investment multiplier explained by the fact that adults borrow to invest in capital considering capital as a collateral. It corresponds to a leverage effect.

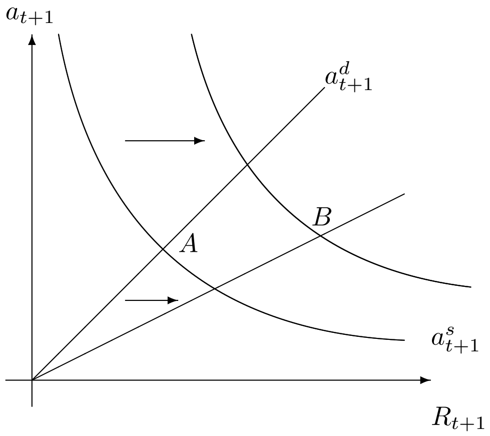

Figure 1: The effect of bubble on capital

Substituting now (21), (22) and (47)-(49) into (42), we get:

$$a_{t+1} = R_{t+1} \frac{\frac{\beta+\gamma}{\alpha+\beta+\gamma} w(a_t) - (x_{1t} + x_{2t})}{\theta(1+\phi)sA + \frac{\alpha}{\alpha+\beta+\gamma}\phi(1-s)A} \equiv a^d_{t+1} \tag{51}$$

This equation is called the asset demand, a^d_{t+1}. It is obtained from the trade-off between consumption when young and adult, and re-expresses the asset demand of young households, which buy the bubble $x_{1t} + x_{2t}$ and the loans collateralized by capital $\theta q_{t+1}(1+\phi)a_{t+1}/R_{t+1}$.

Taking into account that q_t is constant and a_t is predetermined, these two equations represent respectively the asset supply a^s_{t+1} and demand a^d_{t+1} as functions of the interest factor R_{t+1}, for a given level of the bubble $x_{1t} + x_{2t}$. As illustrated in Figure 1, they allow us to understand the effect of the bubble on capital under a binding borrowing constraint.[5] Note that using (50), total savings at time t, $(1+\phi)a_{t+1} + x_{1t} + x_{2t}$, positively depends on $x_{1t} + x_{2t}$, which differs from the model without binding constraint (see equation (40)). This means that following an increase in the level of the bubble $x_{1t} + x_{2t}$, the asset supply increases, while the asset demand decreases, all other things equal (see equations (50 and (51)). Accordingly, a higher level of the bubble may be in accordance with an increase of capital. The bubble may have a crowding-in effect on capital.

Consider first that $b_{2t} = 0$ and $b_{1t} > 0$. Following Farhi and Tirole (2012), we identify the positive effect of b_{1t} on a^s_{t+1} as a liquidity effect. The bubble

[5]Farhi and Tirole (2012) use such a methodology to highlight the effect of the bubble in their framework.

is sold by the adults to the young households, which corresponds to a liquidity transfer from the unproductive young households to the productive adults. The resulting increase in the interest R_{t+1} reduces the investment multiplier $1/(1-\theta q_{t+1}/R_{t+1})$. This effect reduces capital. The net effect on investment of the bubble depends on the interplay between the liquidity effect and the multiplier effect.

When $b_{1t}=0$ and $b_{2t}>0$, we have the same investment multiplier because bubble and credit are perfectly substitutable assets and the borrowing constraint is binding. This explains that the investment multiplier in both the YS and AI models are identical. The other important aspect is to note that credit used by adults to finance capital raises with the bubble and capital, because of the existence of both bubbly and fundamental collaterals. Therefore, the loans d_{1t} provided by young increases with the bubbly and fundamental collaterals. This explains that when adult, the bubble size has a positive effect on a^s_{t+1}, which plays exactly the same role than the liquidity effect identified above.

To summarize, in the model YS, the liquidity effect is due to the savings of young households, part of which is devoted to buy the bubble. In the model AI, the collateral role of the bubble raises the savings of young households through the loans provided. Our analysis does not only show that both mechanisms determine the same equilibrium and will lead to the same result, but that they belong to a more general formulation according to which the equilibrium conditions do not depend on either b_{1t} or b_{2t}, but on the value of $b_{1t}+b_{2t}$.

Using the two equilibrium conditions (50) and (51), we get:

$$\begin{aligned}\frac{\theta q_{t+1}}{R_{t+1}} &= 1-\frac{\gamma}{\beta+\gamma}\frac{\phi w(a_t)+x_{1t}+x_{2t}+\theta q_t(1+\phi)a_t}{(1+\phi)a_{t+1}} \\ &= \frac{\frac{\beta+\gamma}{\alpha+\beta+\gamma}w(a_t)-x_{1t}-x_{2t}}{(1+\phi)a_{t+1}+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi w(a_{t+1})}{\theta q_{t+1}}}\end{aligned}\tag{52}$$

which means

$$\begin{aligned}(1+\phi)a_{t+1} &= \frac{(1+\phi)a_{t+1}}{(1+\phi)a_{t+1}+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi w(a_{t+1})}{\theta q_{t+1}}}\left[\frac{\beta+\gamma}{\alpha+\beta+\gamma}w(a_t)-x_{1t}-x_{2t}\right] \\ &\quad +\frac{\gamma}{\beta+\gamma}\left[\phi w(a_t)+x_{1t}+x_{2t}+\theta q_t(1+\phi)a_t\right]\end{aligned}\tag{53}$$

An equilibrium satisfies (28) and (53), taking into account that R_{t+1} is defined by (52) and the borrowing constraint is binding for $q_{t+1}>R_{t+1}>\theta q_{t+1}$.

By direct inspection of equation (53), we see that if either $\alpha=0$ or $\phi=0$, we get a negative relationship between investment in capital (per unit of labor) a_{t+1} and the bubble size $x_{1t}+x_{2t}$. The crowding-out effect of the bubble dominates. When there is either no consumption when young or no labor income when adult, there is a high incentive when young to transfer purchasing power to the next periods of life to postpone consumption using the bubble and deposits. Any increase in the bubble implies a strong decrease in the deposits used to finance loans collateralized by capital $\theta q_{t+1}(1+\phi)a_{t+1}/R_{t+1}$. It implies, in particular, a strong increase in the interest, which has a negative effect on the investment multiplier, and therefore on capital.

When $\alpha > 0$ and $\phi > 0$, this is no more always the case. Indeed, when α is high, the saving rate of young households is low and, when ϕ is high, the savings of young households are low because the labor income when adult is high. In these last cases, the decrease due to the bubble in the deposits used to finance loans collateralized by capital is limited, making the crowding-in effect of the bubble possible.

3.6 Crowding-in versus crowding-out effect of the bubble

We first analyze a configuration where both capital and the bubble are used as collateral. Then, we study the particular case where the bubble is the only collateral. We end this section giving an intuition for the crowding-in effect of the bubble.

3.6.1 Capital plays the role of collateral, $\theta > 0$

Since we consider an endogenous growth framework, let us define $g_{t+1} = a_{t+1}/a_t$ and $b_t = (x_{1t} + x_{2t})/[(1+\phi)a_t]$. Using (21) and (22), equations (28) and (53) rewrite:

$$g_{t+1}b_{t+1} = R_{t+1}b_t \tag{54}$$

$$\begin{aligned}
&g_{t+1}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right] = \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} \\
&+\frac{\gamma}{\beta+\gamma}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\left[\frac{\phi(1-s)A}{1+\phi}+\theta sA\right] \\
&+b_t\left[\frac{\gamma}{\beta+\gamma}\left(1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right)-1\right]
\end{aligned} \tag{55}$$

which defines $g_{t+1} = F(b_t)$. Using (52), the interest factor is given by:

$$R_{t+1} = \theta s A g_{t+1}\frac{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}-b_t} \equiv R(g_{t+1}, b_t) \tag{56}$$

Substituting (56) into (54), we get the dynamic equation:

$$b_{t+1} = \theta s A\frac{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}-b_t}b_t \equiv G(b_t) \tag{57}$$

with $b_t < \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}$. This equation gives the dynamics of the ratio of bubble over capital, which is a non-predetermined variable. Given the sequence of $(b_t)_{t\geqslant 0}$, we deduce the growth factor at each period of time using (55).

There exist two BGPs, the bubbleless one $(\underline{b}, \underline{g}) = (0, F(0))$ and the bubbly one $(\overline{b}, \overline{g}) = (\frac{\beta+\gamma-\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - \theta s A, F(\overline{b}))$, where $\overline{b} > 0$ if:

$$\phi < \frac{\beta+\gamma}{\alpha} \quad \text{and} \quad \theta < \frac{\beta+\gamma-\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)}{s(1+\phi)} \equiv \theta_a \tag{58}$$

and

$$\overline{g} = A\left[s\theta + (1-s)\frac{\gamma}{\alpha+\beta+\gamma}\right] \tag{59}$$

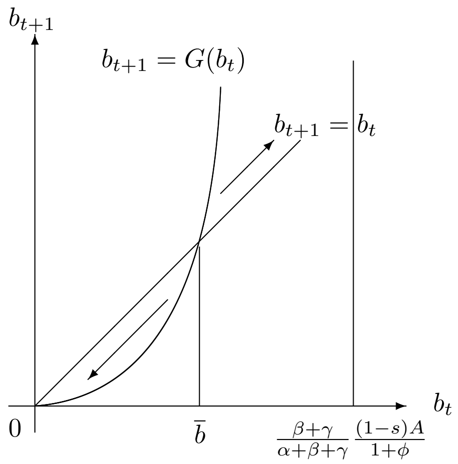

Figure 2: Dynamics of bubble

Taking into account conditions (58), we can easily prove that the bubbleless steady state $\underline{b}$ is stable and the bubbly steady state $\overline{b}$ is unstable. Therefore, there are three types of equilibria depending on agents' expectations:

- there is no bubble, $b_t = \underline{b} = 0$;
- there is a persistent bubble, $b_t = \overline{b} > 0$;
- there is a bubble that decreases and converges to 0 for all $0 < b_t < \overline{b}$.

If individuals initially choose b_t such that $\overline{b} < b_t < \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}$. Along this equilibrium, the bubble would increase and it would eventually crash after a finite number of periods when b_t crosses the upper bound $\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}$. Therefore, rational individuals will never buy the bubble at such a price and, hence, this is never an equilibrium. This means that all equilibria must satisfy $0 \leqslant b_t \leqslant \bar{b}$.[6]

Of course, these equilibria should satisfy the binding borrowing constraint, *i.e.* $q_{t+1} > R_{t+1} > \theta q_{t+1}$.

Lemma 1 *Any equilibrium $b_t \in [0, \bar{b}]$ satisfies the binding borrowing constraint if $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \theta_b$, with:*

$$\theta_b \equiv \alpha\phi \frac{1 - \frac{1-s}{s}\frac{\gamma}{\alpha+\beta+\gamma}}{\alpha\phi + \gamma(1+\phi)} \tag{60}$$

Proof. See Appendix B. ■

Using this lemma, we deduce the existence of the different types of equilibria with bubble:

Proposition 2 *Assuming $\phi < \frac{\beta+\gamma}{\alpha}$, $s > (1-s)\frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \min\{\theta_a, \theta_b\}$, there exist three types of equilibria:*

1. *a bubbleless BGP $b = \underline{b} = 0$;*
2. *a bubbly BGP $b = \bar{b}$;*
3. *any sequence $b_t \in (0, \bar{b})$ which decreases and converges to 0.*

The bubble has a crowding-in (positive) effect on growth if and only if there is a positive relationship between b_t and g_{t+1}. The crowding-out effect dominates when the bubble decreases growth. The following proposition summarizes the main results:

Proposition 3 *Let*

$$\widehat{\theta} \equiv \frac{\gamma}{\beta}\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)} \tag{61}$$

1. *If we have $\phi < \frac{\beta}{\alpha}$ and $s > (1-s)\frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma}$, the bubble has a crowding-in effect on growth if $\theta < \widehat{\theta}$, has no effect on growth if $\theta = \widehat{\theta}$ and has a crowding-out effect on growth if $\widehat{\theta} < \theta < \min\{\theta_a, \theta_b\}$.*[7]
2. *If either $\frac{\beta}{\alpha} \leqslant \phi < \frac{\beta+\gamma}{\alpha}$ or $(1-s)\frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma} \geqslant s > (1-s)\frac{\gamma}{\alpha+\beta+\gamma}$, the bubble has a crowding-in effect on growth for all $\theta < \min\{\theta_a, \theta_b\}$.*

[6] Note that if the bubbly BGP $\bar{b}$ does not exist, *i.e.* $\phi > \frac{\beta+\gamma}{\alpha}$ or $\theta > \theta_a$, we can easily show, using the same arguments than above, that the only equilibrium is the bubbleless BGP.

[7] Notet that $\phi < \frac{\beta}{\alpha}$ and $s > (1-s)\frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma}$ ensure that $\widehat{\theta} < \min\{\theta_a, \theta_b\}$.

Proof. See Appendix C. ■

This proposition shows that when $\theta < \widehat{\theta}$, the existence of the bubble implies higher growth. In this case, the crowding-in effect of the bubble dominates its crowding-out effect. As a direct implication, the bubbly BGP features a higher growth than the bubbleless one, *i.e.* $\overline{g} > \underline{g}$.[8] In contrast, when $\theta > \widehat{\theta}$, the crowding-out effect of the bubble dominates its crowding-in effect and $\overline{g} < \underline{g}$.

Note that this result can be related to Hirano and Yanagawa (2017) who also analyzes an endogenous growth model, but with heterogeneous infinitely-lived agents. In their framework, there is also a level of θ such that below it, the crowding-in effect dominates, whereas above it, the bubble has a crowding-out effect. However, in contrast to us, the existence of the bubble requires a minimum value for θ. The main difference is that in their framework, all investment opportunities have a positive return, whereas in our model, young agents do not invest in capital because they expect a zero return.

It is important to note that, because of our formulation of the bubble, these results hold whatever the type of bubble considered, *i.e.* either young agents buy the bubble ($b_{1t} > 0$), or rather adults ($b_{2t} > 0$), or both. Our results even hold if some agents are short- sellers of the bubble ($b_{1t} < 0$ or $b_{2t} < 0$).

Proposition 3 suggests that a lower θ reinforces the crowding-in effect of the bubble. Using (55), we are able to evaluate the significance of the parameter θ on the level of the crowding-in effect:

$$\frac{dg_{t+1}}{db_t} = \frac{\frac{\gamma}{\beta+\gamma}\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)} - \frac{\beta}{\beta+\gamma}}{1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}}$$

which is decreasing with θ. This means that the crowding-in effect is the larger when $\theta = 0$. We investigate now this particular case.

3.6.2 The limit case where capital has no collateral role, $\theta = 0$

When $\theta = 0$, the borrowing constraint is $-R^d_{t+2}d_{2t+1} \leqslant R_{t+2}b_{2t+1}$, as in Kocherlakota (2009). Taking into account that deposits and bubble are substitutable assets, it rewrites:

$$R_{t+2}x_{2t+1} \geqslant 0 \tag{62}$$

Note that this constraint is of course equivalent to $x_{2t+1} \geqslant 0$ or $-d_{2t+1} \leqslant b_{2t+1}$. This corresponds to a down-payment constraint. Since capital does no more serve as collateral, this case where $\theta = 0$ depicts the more significant degree of financial imperfection.

Whith $\theta = 0$, the equilibrium equations (50) and (51) become (see also

[8] We also observe that the interest factor at the bubbleless BGP, $\underline{R}$, is lower than at the bubbly BGP, $\overline{R}$.

Figure 3):

$$a_{t+1} = \frac{\gamma}{(1+\phi)(\beta+\gamma)}\left[\phi w(a_t) + x_{1t} + x_{2t}\right] \equiv a^s_{t+1} \tag{63}$$

$$a_{t+1} = R_{t+1}\frac{\frac{\beta+\gamma}{\alpha+\beta+\gamma}(1-s)Aa_t - (x_{1t}+x_{2t})}{\frac{\alpha}{\alpha+\beta+\gamma}\phi(1-s)A} \equiv a^d_{t+1} \tag{64}$$

where the asset market clearing satisfies (28).

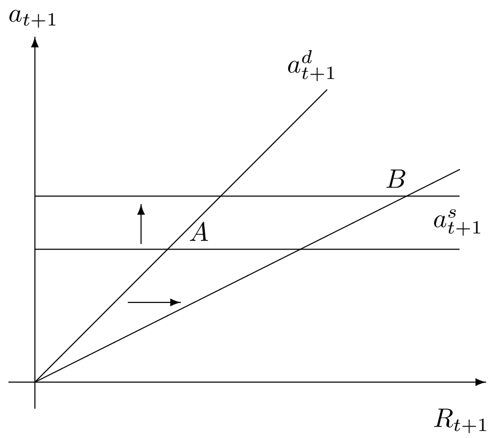

Figure 3: The effect of bubble on capital when $\theta = 0$

Using (28), (63), (64) and $b_t = \frac{x_{1t}+x_{2t}}{(1+\phi)a_t}$, an equilibrium is defined by a sequence (b_t, g_{t+1}), satisfying:

$$g_{t+1} = \frac{\gamma}{\beta+\gamma}\left[\frac{\phi(1-s)A}{1+\phi} + b_t\right] \tag{65}$$

$$b_{t+1}g_{t+1} = R_{t+1}b_t \tag{66}$$

where

$$R_{t+1} = \frac{\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)A}{1+\phi}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t}g_{t+1} \tag{67}$$

By direct inspection of equation (65), we see that the bubble has a crowding-in effect on economic growth. Substituting (67) into (66), the dynamics are driven by:

$$b_{t+1} = \frac{\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)A}{1+\phi}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t}b_t \tag{68}$$

We deduce that the analysis is quite similar than in the previous models. In particular, there is a bubbleless BGP $b = 0$ and a bubbly one:

$$\bar{b} = \frac{\beta + \gamma - \alpha\phi}{\alpha + \beta + \gamma} \frac{(1-s)A}{1+\phi} > 0 \text{ if } \frac{\beta + \gamma}{\alpha} > \phi \tag{69}$$

Using (65) and (66), this BGP is characterized by $\overline{R} = \overline{g} = \frac{\gamma(1-s)A}{\alpha+\beta+\gamma} > 0$. Finally, the borrowing constraint is binding if $\overline{R} < sA$, which is equivalent to $s > (1-s)\frac{\gamma}{\alpha+\beta+\gamma}$.

Corollary 1 *Assume $\theta = 0$. If $\phi < \frac{\beta+\gamma}{\alpha}$ and $s > (1-s)\frac{\gamma}{\alpha+\beta+\gamma}$, the bubbly BGP exists. In addition, the bubble has a crowding-in effect on growth.*

This corollary shows that the bubble has a positive effect on growth when capital does not play the role of collateral. Therefore, there is a higher growth at the bubbly than at the bubbleless BGP.

3.6.3 Economic interpretation

We would like first to understand why the bubble always has a crowding-out effect when there is no binding borrowing constraint, while the existence of a bubble can promote growth when the investors are constrained. When there is no binding constraint, households can perfectly smooth consumption and, hence, savings do not depend on the bubble. In this case, consumptions linearly depend on the life-cycle income (see (30)-(32)), which implies that total savings depend on a weighted sum of discounted wages (see (40)). Moreover, all assets are perfect substitutes, *i.e.* have the same return. As a direct implication, any increase of the bubble implies a decrease of the new investment in capital.

When adults face a binding borrowing constraint, households can no more smooth consumption without any restrictions and all assets are no more substitutes. This means that the consumptions depend now on the asset holdings and, therefore, the savings too. This opens the door to mechanisms for which the bubble has a crowding-in effect on capital.

As the borrowing constraint is binding, adults use the bubble and credit to finance capital, meaning that $x_{2t} = b_{2t} + d_{2t}$ is negative. In a way, they transfer purchasing power from the old age. In contrast, young households use deposit and the bubble to postpone consumption from the first period of life, explaining that $x_{1t} = b_{1t} + d_{1t}$ is positive. This allows households to increase investment, because adults have more liquidities to buy capital (liquidity effect). This can also be interpreted as a transfer from the unproductive young traders to the productive adult ones. Both these effects enhance investment in capital. Note that these effects could be achieved with either the bubble or with credit. In what follows, we explain the specific effects of the bubble.

By direct inspection of equation (49), we first observe that the consumption when old does not directly depend on the level of the bubble because of the binding borrowing constraint. The redistribution from the old to the adult age only depends on capital income, because the demand of loans net of the purchase or sale of the bubble is constrained by the fundamental collateral. If we focus now on consumptions when young and adult, given by (47) and (48), we easily see that the bubble results in a redistribution from the young households to the adults. If the bubble is bought when young ($b_{1t} > 0$), we have the standard liquidity effect and the explanation is straightforward. This is the specific effect introduced by the bubble and that may cause a crowding-in effect. If the bubble is bought when adult ($b_{2t} > 0$), the argument goes through the credit. Indeed, through the borrowing constraint, a higher bubble means more loans, which requires more deposits by young households. It increases the liquidity transferred at the adult age. Since deposits, or credit, and bubbles are perfectly substitutable assets, the liquidity role of both types of bubbles (b_{1t} or b_{2t}) are identical. Our approach of bubbles allows us to deduce that what is important is not to know whether b_{1t} or b_{2t} is positive, negative or zero, but to know the level of $b_{1t} + b_{2t}$. Any combinations of b_{1t} and b_{2t} that keep $b_{1t} + b_{2t}$ constant give the same result. Of course, credit cannot have such a role, because deposits are entirely used to finance loans, meaning that $d_{1t} + d_{2t} = 0$, while the bubble has a positive value, $b_{1t} + b_{2t} > 0$.

Finally, the existence of a bubble, which means a higher supply of liquid assets, also increases the cost of the credit used to finance capital (R_{t+1} increases), which reduces capital investment. Therefore, the bubble enhances growth when the degree of pledgeability θ is sufficiently small, because as it is clear from the asset market described by equations (50) and (51), the higher θ, the more important the negative effect of a raise of R_{t+1} on capital, due to adults' demand of credit. This last effect corresponds to the crowding-out effect of the bubble when the borrowing constraint is binding.

To further discuss the role of the degree of pledgeability θ, we easily see that both growth factors $\overline{g}$ and $\underline{g}$ evaluated respectively at the bubbly and bubbleless BGP increase with θ (see equations (55) and (59)). Indeed, a higher θ means a higher role for the fundamental collateral and, therefore, higher borrowing to finance capital investment. Since the crowding-in effect of the bubble dominates for $\theta < \widehat{\theta}$ and the crowding-out effect dominates for $\theta > \widehat{\theta}$, there is a positive gap between $\overline{g}$ and $\underline{g}$ when $\theta = 0$, which decreases and becomes negative when θ becomes higher than $\widehat{\theta}$. This means that when $\theta = 0$, the gap between $\overline{g}$ and $\underline{g}$ is the highest one, but the growth rates are lower than the growth rates for higher values of θ.

4 Concluding remarks

Recently, several papers have identified some channels through which asset bubbles promote economic activity, as it is empirically observed. One important feature is the existence of some borrowing constraints, and another one is the heterogeneity of traders' behavior. Bubbles can have different roles. Two main are to provide liquidities and to serve as collateral. In this paper, we introduce heterogeneous traders by considering an overlapping generations model with three period-lived households. Only adults have access to capital investment, and face a borrowing constraint. We show that the roles played by a bubble, namely to provide liquidities and be a collateral, are perfectly equivalent. We introduce asset bubbles in an even more general way, that encompasses the roles just mentioned but not only, and show that a bubble may enhance growth. This conclusion is true for a given value of the bubble, whatever the type and the role attributed to the bubble, and who holds this asset.

As we have seen, the asset bubble and the credit market allow to make some transfers from the young and old agents to adults who invest in capital. Of course, if the young agents invest in capital rather than the adults, the conclusions are completely different. As shown by Raurich and Seegmuller (2019), the transfers are done from the adult age to the young and old ones. In this paper, the bubble enhances production because its existence relaxes the binding credit constraint and facilitates investment. Our paper is complementary to this previous one. Using these two contributions, the conditions to have a crowding-in effect of the bubble are established, regardless the investment in capital in done at the young or at the adult age.

Appendix

A Proof of Proposition 1

Define $\widetilde{\Omega}(g) \equiv \Omega(g, g)$. Using (38), we note that $\widetilde{\Omega}'(g) < 0$. We deduce that, at a BGP, $db/dg < 0$.

Consider now equation (35) at a steady state. When there is a bubble, the growth factor is given by $g = sA \equiv g^*$ and the bubble is positive if $\widetilde{\Omega}(g^*) > 0$, which is equivalent to (39).

At a bubbleless BGP, $b = 0$ and the associated growth factor g^{**} solves $\widetilde{\Omega}(g^{**}) = 0$. Since $\widetilde{\Omega}'(g) < 0$, we easily conclude that $g^{**} > sA = g^*$.

To analyze the stability properties of these two steady states, we substitute (38) in (35) to obtain:

$$\Omega(g_{t+1}, g_{t+2})g_{t+1} = sA\Omega(g_t, g_{t+1})$$

which is equivalent to:

$$\begin{aligned} & g_{t+2}\left[1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{(1+\phi)s}\right] + \frac{1}{g_{t+1}}\left[\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{1+\phi} + \frac{1}{g_t}\frac{\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{1+\phi}\right] \\ & -sA\left[1 + \frac{1-s}{s(1+\phi)}\frac{\alpha\phi+\beta+\gamma(1+\phi)}{\alpha+\beta+\gamma}\right] = 0 \end{aligned}$$

Linearizing this equation in the neighborhood of a steady state g, we get the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$, where the trace T and the determinant D of the associated Jacobian matrix are given by:

$$T = \frac{\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{1+\phi} + \frac{\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{(1+\phi)g}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}}\frac{1}{g^2} > 0 \tag{A.1}$$

$$D = -\frac{\frac{\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{(1+\phi)g^3}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} < 0 \tag{A.2}$$

Using these two equations, we easily compute:

$$P(-1) = 1+T+D = 1+\frac{\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{(1+\phi)g^2}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} > 0, \text{ for all } g > 0 \tag{A.3}$$

We now determine:

$$P(1) = 1-T+D = 1-\frac{\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma}\frac{s(1-s)A^2}{(1+\phi)g^2} + \frac{2\gamma}{\alpha+\beta+\gamma}\frac{s^2(1-s)A^3}{(1+\phi)g^3}}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} \tag{A.4}$$

At the steady state $g = g^* = sA$,

$$P(1) = 1 - \frac{(1-s)[\beta+\gamma(2+\phi)]}{\alpha(s+\phi)+(\beta+\gamma)s(1+\phi)} < 0 \tag{A.5}$$

under inequality (39). Since $P(-1) > 0$, $P(0) = D < 0$, $P(1) < 0$ and $P(+\infty) > 0$, the bubbly steady state is a saddle because the two eigenvalues satisfy $\lambda_1 > 1$ and $\lambda_2 \in (-1, 0)$.

Using $\widetilde{\Omega}(g^{**}) = 0$, equation (A.4) evaluated at the steady state $g = g^{**}$ also writes:

$$P(1) = \frac{\left(\frac{\beta+\gamma\phi}{\alpha+\beta+\gamma} + \frac{\gamma}{\alpha+\beta+\gamma}\frac{sA}{g^{**}}\right)\frac{(1-s)A}{(1+\phi)g^{**}}\left(1-\frac{sA}{g^{**}}\right) + \frac{\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{(1+\phi)g^{**}}\left(1-\frac{s^2A^2}{g^{**2}}\right)}{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}} \tag{A.6}$$

Since $g^{**} > sA$, we deduce that $P(1) > 0$. Therefore, at this steady state, we have $P(-1) > 0$, $P(0) < 0$ and $P(1) > 0$, meaning that the eigenvalues are such that $\lambda_1 \in (0, 1)$ and $\lambda_2 \in (-1, 0)$.

B Proof of Lemma 1

Using (20) and (56), $R_{t+1} > \theta q_{t+1}$ is equivalent to:

$$g_{t+1}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right] > \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t \tag{B.7}$$

By inspection of (55), this inequality is always satisfied. Using again (20) and (56), $R_{t+1} < q_{t+1}$ is equivalent to:

$$RHS(b_t) \equiv \frac{\theta\gamma}{\beta+\gamma}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\left[b_t+\frac{\phi(1-s)A}{1+\phi}+\theta sA\right]$$
$$< (1-\theta)\left[\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}-b_t\right] \equiv LHS(b_t) \quad \text{(B.8)}$$

For $b_t \leqslant \bar{b}$, we deduce that:

$$\begin{aligned} RHS(b_t) &\leqslant \left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\frac{(1-s)A\theta\gamma}{\alpha+\beta+\gamma} \\ LHS(b_t) &> (1-\theta)\frac{\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} \end{aligned}$$

Using these last two inequalities, inequality (B.8) is satisfied if $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \theta_b$.

C Proof of Proposition 3

Assume $\phi < \frac{\beta+\gamma}{\alpha}$, $s > (1-s)\frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \min\{\theta_a, \theta_b\}$. Using (55), g_{t+1} is increasing (decreasing) in b_t if and only if $\theta < \widehat{\theta}$ ($\theta > \widehat{\theta}$) and g_{t+1} does not depend on b_t if and only if $\theta = \widehat{\theta}$. Then, using (58) and (61), we can show that $\widehat{\theta} < \theta_a$ is equivalent to:

$$\phi < \frac{\beta}{\alpha} < \frac{\beta+\gamma}{\alpha}$$

Using now (60) and (61), $\widehat{\theta} < \theta_b$ if and only if:

$$\phi[\gamma(1-s)-\beta s] < \beta s-(1-s)\gamma\frac{\beta+\gamma}{\alpha+\beta+\gamma}$$

which is equivalent to $s > (1-s)\frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma} > (1-s)\frac{\gamma}{\alpha+\beta+\gamma}$. Using Proposition 2, we easily deduce the proposition.

D Introduction of a degree of pledgeability for the bubbly collateral

Let us consider that the borrowing constraint is now characterized by a degree of pledgeability $\xi \in (0,1)$ for the bubbly collateral. This means that the borrowing constraint rewrites:

$$-R^d_{t+2}d_{2t+1} \leqslant \theta q_{t+2}k_{t+2} + \xi R_{t+2}b_{2t+1} \quad \text{(D.9)}$$

Except this constraint, the model is similar than in Section 3.3. We associate the multiplicators λ_{1t}, λ_{2t+1} and λ_{3t+2} to the budget constraints (23), (24) and

(25), and μ_{t+1} to the borrowing constraint (D.9). We obtain the following first order conditions:

$$\alpha u_1'(c_{1t}) = \lambda_{1t}, \beta u_2'(c_{2t+1}) = \lambda_{2t+1}, \gamma u_3'(c_{3t+2}) = \lambda_{3t+2} \tag{D.10}$$
$$\lambda_{1t} = R^d_{t+1}\lambda_{2t+1}, \lambda_{1t} = r_{t+1}\lambda_{2t+1} \tag{D.11}$$
$$\lambda_{2t+1} = q_{t+2}(\lambda_{3t+2} + \theta\mu_{t+1}) \tag{D.12}$$
$$\lambda_{2t+1} = R^d_{t+2}(\lambda_{3t+2} + \mu_{t+1}) \tag{D.13}$$
$$\lambda_{2t+1} = R_{t+2}(\lambda_{3t+2} + \xi\mu_{t+1}) \tag{D.14}$$

Using (D.11), we easily deduce that $R^d_{t+1} = r_{t+1}$. Using (D.12)-(D.14), we get $R^d_{t+2} < q_{t+2}$ and $R^d_{t+2} < R_{t+2}$.

We deduce that at an equilibrium, $r_{t+1} = R^d_{t+1} < R_{t+1}$. This means that b_{1t} and b_{2t} cannot represent the same asset, but are different bubbly assets with different returns:

$$b_{1t+1} = r_{t+1}b_{1t} \tag{D.15}$$
$$b_{2t+1} = R_{t+1}b_{2t} \tag{D.16}$$

Let us introduce $\widehat{b}_{it} \equiv b_{it}/[(1+\phi)a_t]$. Equations (D.15) and (D.16) are equivalent to:

$$g_{t+1}\widehat{b}_{1t+1} = r_{t+1}\widehat{b}_{1t} \tag{D.17}$$
$$g_{t+1}\widehat{b}_{2t+1} = R_{t+1}\widehat{b}_{2t} \tag{D.18}$$

An equilibrium with $b_{1t} > 0$ and $b_{2t} > 0$ requires that $r_{t+1} \leqslant g_{t+1}$ and $R_{t+1} \leqslant g_{t+1}$ for an infinite number of periods.[9] Otherwise, one of the bubbly asset explodes and can no more be bought by the households. In such a case, we have $b_{1t} = 0$ and/or $b_{2t} = 0$, which rules out the liquidity and/or the collateral role of the bubble. Assuming that the last two inequalities hold, we have:

$$\frac{\widehat{b}_{1t+1}}{\widehat{b}_{2t+1}} = \frac{r_{t+1}}{R_{t+1}}\frac{\widehat{b}_{1t}}{\widehat{b}_{2t}} \tag{D.19}$$

Since $r_{t+1} < R_{t+1}$, $\widehat{b}_{1t}/\widehat{b}_{2t}$ tends to zero in the long run. This means that with respect to the collateral role, the liquidity role of the bubble disappears in the long run.

The reason of this result is the following. The bubbly asset bought when young is a perfect substitute to the debt. Therefore, both these assets have the same return. Because of the degree of pledgeability $\xi < 1$ in the borrowing constraint (D.9), the bubble bought when adult is less useful than debt. This implies that the return R_{t+2} of b_{2t+1} need to be higher than the return of debt, otherwise this asset will never be hold by households. This implies that the bubbly asset bought at the adult age b_{2t} grows at a higher rate than b_{1t}, and the liquidity role of the bubble tends to disappear in the long run.

[9]Both r_{t+1} and R_{t+1} can be strictly higher than g_{t+1} for a finite number of periods.

Note that, in the model, we do not impose that b_{1t} and b_{2t} represent a priori the same asset. If it was the case, they would have the same return $r_{t+1} = R_{t+1}$. Using (D.11), the first order condition (D.13) would hold has an equality, but (D.14) would become the inequailty $\lambda_{2t+1} > R_{t+2}(\lambda_{3t+2} + \xi\mu_{t+1})$. This would imply that households sell short an infinite amount of the bubbly asset b_{2t}, which cannot be sustained as an equilibrium.

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