Disastrous Defaults*

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Abstract

We define a *disastrous default* as the default of a systemic entity, which has a negative effect on the economy and is contagious. Bringing macroeconomic structure to a no-arbitrage asset-pricing framework, we exploit prices of disaster-exposed assets (credit and equity derivatives) to extract information on the expected (i) influence of a disastrous default on consumption and (ii) probability of a financial meltdown. We find that the returns of disaster-exposed assets are consistent with a systemic default being followed by a 3% decrease in consumption. The recessionary influence of disastrous defaults implies that financial instruments whose payoffs are exposed to such credit events carry substantial risk premiums. We also produce systemic risk indicators based on the probability of observing a certain number of systemic defaults or a sharp drop of consumption.

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1 Introduction

Since the seminal contribution of Rietz (1988), studies have shown that disaster risk, defined as a sudden and dramatic decrease in output and consumption, helps in solving several asset pricing puzzles (e.g. Barro, 2006; Gabaix, 2012; Gourio, 2013). In these studies, disasters are typically modeled as exogenous events that cause dramatic increases in the default probabilities of bond issuers (or dramatic decreases in the asset values of firms). However, recent events suggest that the default of a systemic entity *per se* may very well constitute a disaster in itself. Indeed, since its inception, the largest drop in the University of Michigan Consumer Sentiment index took place in September 2008, the month when Lehman Brothers went bankrupt. By the same token, the existence of systemic entities is at the core of novel regulations on Systemically Important Financial Institutions – SIFIs (International Monetary Fund, 2010; Basel Committee on Banking Supervision, 2013; Battiston et al., 2016; Kelly et al., 2016; Brownlees and Engle, 2017).

In this paper, we propose an asset pricing framework where the default of some entities – called systemic – may have disastrous economic effects. In the model, the default of each systemic entity can affect consumption; moreover such an event can be the source of default cascades.[1] When they materialize, new defaults are likely to reinforce the initial drop in consumption and to contribute to further defaults. The model therefore accommodates amplification mechanisms (Allen and Gale, 2000; Stiglitz, 2011). In this context, financial instruments that are exposed to the default of systemic entities are expected to command substantial risk premiums; the latter being defined as the component of prices that would not exist if agents' were risk-neutral.

To our knowledge, the present study constitutes the first attempt to measure the macroeconomic influence of contagious corporate defaults. This information is extracted from the joint dynamics of consumption and of the prices of disaster-exposed market instruments. The model tractability, which is instrumental for our study, hinges on the "top-down" approach used to model the defaults of systemic entities. Top-down credit-risk models focus on default counting processes; in contrast to the "bottom-up" approach that considers default processes of individual firms as the model

[1] Das et al. (2007) find that default clustering cannot be explained only by the firms' joint exposure to observable systematic factors. In a recent paper, Azizpour et al. (2018) provide strong evidence for the fact that contagion, through which the default of one firm has a direct impact on the health of other firms, is a significant clustering source.

primitives. While the default indicators of all contagious firms have to be included in the state vector to account for contagion in the bottom-up approach, keeping track of the number of systemic defaults is sufficient in top-down models. The direct modeling of the default counting process allows to easily accommodate two important phenomena: (i) contagion, which simply amounts to making the default counting process self-excited (e.g. Giesecke and Kim, 2011; Giesecke et al., 2011), and (ii) the recessionary influence of defaults, by making consumption growth depend on the number of systemic defaults. These two features are not present in the bottom-up frameworks proposed by Collin-Dufresne et al. (2012), Christoffersen et al. (2017), or Seo and Wachter (2018). Moreover, contrary to the present framework, bottom-up-based studies have to rely on computer-demanding simulations to price credit derivatives whose payoffs depend on the number of defaults, making their estimation particularly challenging.[2]

We estimate our model by exploiting market data on two types of financial instruments that are directly exposed to systemic risk: tranches of synthetic Collateralized Debt Obligations and far-out-of-the-money put options written on equity indexes. In synthetic CDO transactions, the protection buyer receives payments when a pre-specified amount of credit losses in the reference portfolio has been reached. Losses are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritized tranches (mezzanine tranches, followed by senior tranches). Therefore, senior tranches provide non-null payoffs to the protection buyer only once a sufficiently large number of entities in the underlying portfolio have defaulted. As a result, market prices of senior tranches reflect investors' expectations regarding catastrophic events (Coval et al., 2007; Longstaff and Rajan, 2008; Collin-Dufresne et al., 2012). Even the senior tranches of CDOs written on a portfolio of investment grade firms display non-negligible prices. This suggests that investors allocate a non-zero probability to events of unseen proportions. The second type of financial instruments, far-out-of-the-money put options, deliver payoffs when the underlying equity index experiences crashes. Typically, if the strike price is equal to 70% of the current value of the equity index, then this option yields a strictly positive payoff when the equity index falls by 30%

[2] Bai et al. (2015) propose a bottom-up framework allowing for contagion and systemic entities – i.e. entities whose defaults affect consumption. These authors focus on the one-period returns of firms asset values. Their empirical analysis does not involve multi-period returns of financial instruments whose payoffs depend on the defaults of several firms, which would be challenging in the context of their framework.

between the inception and the maturity date of the contract. Therefore, such equity options also convey information regarding the market perception of systemic risk (Santa-Clara and Yan, 2010; Backus et al., 2011).

The empirical application, which is conducted on euro-area data spanning the period from January 2006 to September 2017, demonstrates the ability of our model to capture a substantial share of the joint fluctuations of consumption growth, stock returns and stock and credit derivatives, both in tranquil and stressed periods.[3] Specifically, our estimation involves prices of derivatives written on (i) the EURO STOXX 50 index, one of the main benchmarks of European equity markets, and (ii) the credit portfolio underlying the iTraxx Europe main index, including synthetic CDOs of different maturities and seniority levels. Our estimation procedure assigns all 125 constituent entities of the iTraxx index – the most liquid European investment grade credits – as systemic.

In the spirit of Backus et al. (2011), we deduce estimates of the influence of systemic defaults on consumption. Our results suggest that the default of a systemic entity is expected to be followed by a 3% decrease in consumption within two years, taking contagion effects into account. Let us provide some intuition as to why this influence can be inferred from our estimation. Our equilibrium model provides some structure regarding risk premiums; specifically, once the relationship between the payoffs of a given asset and the factors affecting consumption has been specified, the model predicts the size of risk premiums asked by investors to carry this asset. Now, the payoffs of a CDS written on a systemic entity critically depend on the default status of this entity. As a result, through the lens of the model, the potential influence of a "systemic default" on consumption can be inferred from the size of credit risk premiums embedded in a CDS spread written on a systemic entity.

We further exploit our estimated model to derive two systemic risk indicators. The first indicator is defined as the probability of observing a certain number of systemic defaults over specific horizons.[4] The resulting systemic indicator reaches its highest levels in late 2008, after the Lehman

[3] Hence, our results contribute to the growing literature investigating the links between stock and fixed-income prices (see e.g. Bekaert et al., 2010; Lustig et al., 2013; Koijen et al., 2017; Campbell et al., 2017).

[4] Using prices of far-out-of-the-money put options to infer disaster probabilities dates back to Bates (1991). This approach has been applied recently by Backus et al. (2011), Bollerslev and Todorov (2011), Barro and Liao (2016), Siriwardane (2016) and Seo and Wachter (2018), among others. For a discussion regarding the difficulty in measuring

bankruptcy and in late 2011, when the European sovereign crisis was at its peak. On these two dates, the probabilities of having at least 10 iTraxx constituents defaulting within two years were of 7% and 6%, respectively. The second indicator is defined as the probability of consumption displaying a sharp drop in the next year. In line with the first indicator, these probabilities reached their maximum levels at the time of the Lehman bankruptcy and at the height of the European sovereign crisis. On these two dates, the probabilities of consumption dropping by more than 10% in the next year were of 6% and 5%, respectively.

Moreover, our findings point to the existence of substantial credit risk premiums in the credit derivatives written on systemic entities. In particular, the results suggest that about two thirds of 10-year Credit Default Swaps (CDSs) spreads written on systemic entities correspond to credit risk premiums. In other words, if agents were not risk-averse, these spreads would be three times lower. In line with previous studies (Brigo et al., 2009; Azizpour et al., 2011; Giesecke and Kim, 2011), we find that an overwhelming share of the prices of the most senior tranches corresponds to risk premiums.

We also analyze the pricing of credit derivatives written on entities that are not systemic. Non-systemic entities are defined as entities: (i) whose default does not cause other defaults, and (ii) that have no macroeconomic impact. These non-systemic entities may however be exposed to systemic defaults – through contagion effects – and/or to other macroeconomic variables. We show that, for a fixed probability of default, the higher the exposure of these entities to systemic defaults, the higher the spreads of CDSs written on these (non-systemic) entities.

The remainder of this paper is organized as follows. Section 2 presents the general framework and derives associated pricing formulas. Section 3 describes the data and Section 4 documents the estimation approach. Section 5 explores the asset pricing and macroeconomic implications of systemic defaults. Section 6 concludes. The derivation of pricing formulas are gathered in appendices.

systemic risk, refer to Hansen (2013).

2 Model

The present model, which builds on Gouriéroux et al. (2014), falls in the category of "top-down" models, which focus on default counting (or loss) processes (e.g. Azizpour et al., 2011; Giesecke et al., 2011). These models contrast the "bottom-up" approach that considers default processes of individual firms as the model primitives (e.g. Lando, 1998; Duffie and Singleton, 1999; Duffie and Gârleanu, 2001).[5]

2.1 Credit segments

We consider J homogeneous segments of defaultable entities. For any j, the I_j entities of segment j share the same credit characteristics.

Let $N_{j,t}$ be the number of segment-j entities in default at date t and N_t be the vector $N_t = [N_{1,t},\ldots,N_{J,t}]'$. We denote by $n_{j,t} = N_{j,t} - N_{j,t-1}$ the number of defaults occurring in segment j on date t. With obvious notations, we have $n_t = N_t - N_{t-1}$.

The information on current and past values of any process k_t is denoted by $\underline{k_t} = \{k_t, k_{t-1}, \ldots\}$. Conditional on $N_0 = 0$, the information contained in the information set $\underline{n_{j,t}}$ (respectively $\underline{n_t}$) is equivalent to that in $\underline{N_{j,t}}$ (resp. $\underline{N_t}$).

The first two segments of entities gather large firms supposed to be systemic. We denote by $N^s_t = N_{1,t} + N_{2,t}$ the cumulated number of systemic defaults and by $n^s_t = n_{1,t} + n_{2,t}$ the number of systemic defaults occurring on date t. The only distinction between these first two segments is that the first one contains the constituents of a credit index used as the reference portfolio for traded credit derivatives, whose prices are used to calibrate the model. Having a single segment of systemic entities would be restrictive because it would mean that this specific credit index, namely the iTraxx Europe main, covers all systemic entities in our economy, which may not be realistic. The other segments, $j \geq 3$, gather non-systemic firms, that can be thought of as small firms.

[5] The "top-down" approach has been shown to satisfactorily capture the existence of default clustering (e.g. Brigo et al., 2007; Errais et al., 2010).

2.2 Default-count processes

We assume that defaults are caused by two exogenous and non-negative factors that we denote by x_t and y_t, which allows to distinguish between longer-run and shorter-run fluctuations of aggregate credit risk. In combination, these factors may summarize business cycle fluctuations. Without loss of generality, we impose $\mathbb{E}(x_t) = \mathbb{E}(y_t) = 1$. Appendix A.1 proposes a specification based on vector auto-regressive gamma processes (see Gouriéroux and Jasiak, 2006; Monfort et al., 2017). These processes are Markovian, with gamma-type transition distributions, and feature conditional heteroscedasticity. In particular, they are such that:

$$\left\{ \begin{array}{rcl} x_t - 1 &=& \rho_x(x_{t-1} - 1) + \sigma_{x,t}\varepsilon_{x,t}, \\ y_t - x_t &=& \rho_y(y_{t-1} - x_{t-1}) + \sigma_{y,t}\varepsilon_{y,t}, \end{array} \right. \quad (1)$$

where $\varepsilon_t = [\varepsilon_{x,t}, \varepsilon_{y,t}]'$ is a martingale difference sequence with unit-variance components and where $[\sigma^2_{x,t}, \sigma^2_{y,t}]'$ is affine in $[x_{t-1}, y_{t-1}]'$.

Intuitively, if $0 < \rho_y < \rho_x < 1$, the autonomous factor x_t is more persistent than $y_t - x_t$ and, if ρ_x is large, x_t can be seen as the low-frequency component of y_t. The residual component $y_t - x_t$, which has a marginal zero expectation, can be interpreted as the higher-frequency component of y_t. The higher y_t, the higher the probability of default of the different entities. Formally, for any segment j, we assume that $n_{j,t}$ is an integer process with stochastic intensity, defined by:

$$n_{j,t+1}|\underline{x_{t+1}}, \underline{y_{t+1}}, \underline{N_t} \quad \sim \quad \mathscr{P}(\beta_j y_{t+1} + c_j n^s_t), \quad (2)$$

where β_j and c_j are non-negative parameters, where n^s_t is the total number of systemic defaults taking place on date t and $\mathscr{P}$ denotes the Poisson distribution. If $c_j > 0$, the occurrence of systemic defaults on date t increases the conditional probability of having defaults in segment j on the next date. Thus, systemic defaults are contagious when $c_j > 0$ for some j.[6,7] By contrast, the defaults of non-systemic segments are not contagious since, for $j > 2$, $n_{j,t}$ does not appear in the parameter

[6]In particular, if $c_j > 0$ for $j \in \{1,2\}$ (systemic segments), then systemic defaults are "self-excited" (Aït-Sahalia et al., 2014).

[7]Systemic defaults are contagious, or "infectious", using Davis and Lo (2001)'s terminology.

of the Poisson distribution in eq. (2).

For parsimony, we consider that entities from the two systemic segments are alike, the only difference being that those from segment 1 are the constituents of traded credit indexes. Accordingly, we assume that $c_1 = c_2$ and that $\beta_1 = \beta_2$. That is, assuming also that $I_1 = I_2$, the conditional default probabilities of a systemic entity, be it of segment 1 or 2, are the same.

Eq. (2) specifies a default process $N_{j,t}$ that does not necessarily terminate at I_j, the number of entities in segment j. As noted by Azizpour et al. (2011), this feature is, however, innocuous because for the relatively large portfolios of interest, the probability of $N_{j,t}$ exceeding I_j during standard contract terms is small for our sample.[8,9]

2.3 Consumption growth process

We assume that systemic defaults have a negative impact on the log growth rate of per capita consumption, denoted by $\Delta c_t = \log(C_t/C_{t-1})$. To have this, a possibility is to make Δc_t directly depend on the number of systemic defaults n^s_{t-1}. However, this would have the unrealistic implication that all systemic defaults have the same deterministic effect on consumption growth. Instead, we assume that Δc_t depends on a factor w_t, that depends itself on systemic defaults according to:

$$w_t | \underline{x_t}, \underline{y_t}, \underline{N_t} \sim \gamma_0(\xi_w n^s_{t-1}, \mu_w), \tag{3}$$

where the zero-gamma distribution γ_0 is a distribution featuring a point mass at zero (Monfort et al., 2017). Specifically, when $n^s_{t-1} > 0$, w_t is drawn from a gamma distribution whose scale parameter is μ_w and whose shape parameter is drawn from $\mathscr{P}(\xi_w n^s_{t-1})$. When the shape parameter is zero, we have $w_t \equiv 0$. Therefore, in this context, the conditional probability that $w_t = 0$ is $\exp(-\xi_w n^s_{t-1})$. In particular, we have $w_t = 0$ as long as there has been no systemic defaults in the previous period. For identification, we impose $\mathbb{E}(w_t) = 1$, which is obtained by setting $\mu_w = 1/(\xi_w \mathbb{E}(n^s_t))$.

[8] Size effects are captured by parameters β_j and c_j. In our empirical study, the segment sizes we use are 50 (EURO STOXX 50) and 125 (iTraxx index), see Subsection 3.1.

[9] The fact that $N_{j,t}$ does not terminate at I_j is important if one wants to use this type of model to produce long-term forecasts; it implicitly entails a form of replacement mechanism of systemic entities.

The consumption growth process is then specified as follows:

$$\Delta c_t = \mu_{c,0} + \mu_{c,x} x_t + \mu_{c,y} y_t + \mu_{c,w} w_t + \sigma_c \varepsilon_t^c \quad \varepsilon_t^c \sim i.i.d.\mathcal{N}(0,1). \tag{4}$$

That is, consumption growth is affected by: the business-cycle components x_t and y_t, the disaster-like component w_t and a volatile normally-distributed shock ε_t^c.

As is standard in this literature, we do not account for inflation in our model. That is, we assume that the inflation rate is constant, or independent from (x_t, y_t, w_t, N_t).

Panel (a) of Figure 1 shows the resulting causal scheme. In our model, the defaults of non-systemic segments ($j > 2$) have no causal impact on consumption or on defaults in other segments. As a result, non-systemic segments are not used in the model estimation. We will however use segment 3 in Section 5 to study the implications of the model for the pricing of credit derivatives written on non-systemic entities.

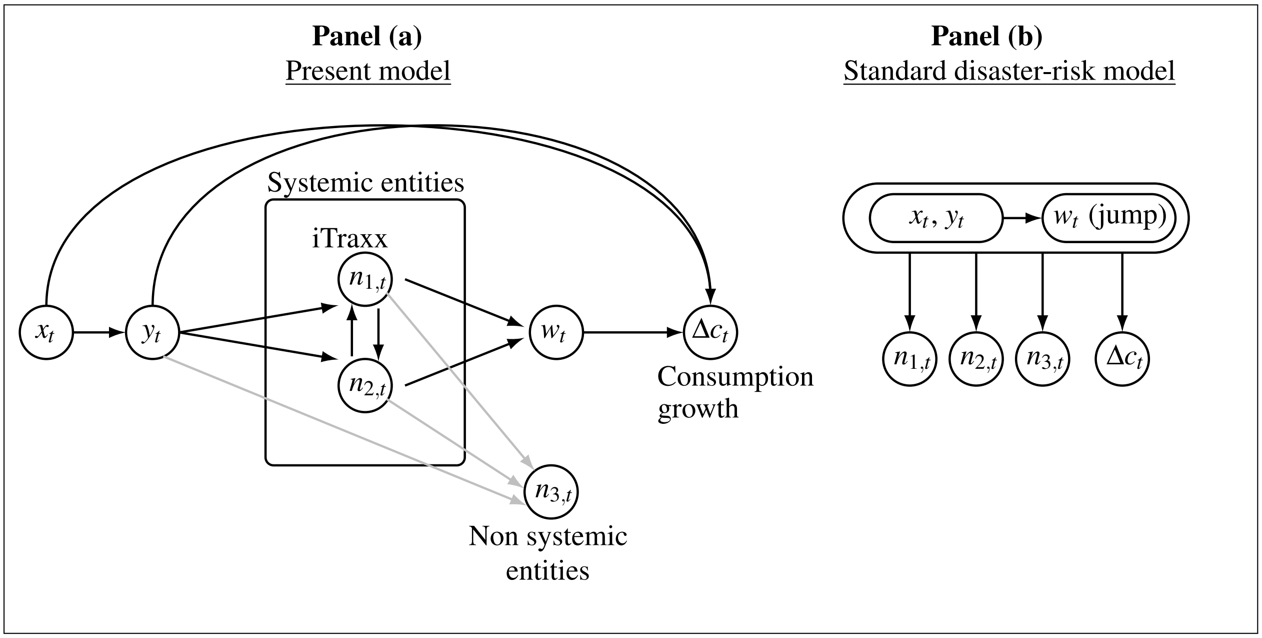
Panel (b) of Figure 1 represents the type of scheme prevailing in standard disaster-risk models. In these models, disasters take the form of jumps that simultaneously trigger a fall in consumption and sharp increases in default probabilities. By contrast, in our context, the defaults themselves cause drops in consumption. Because this mechanism is the focus of the present study, we do not incorporate feedback effects from consumption to factors x_t and y_t.

2.4 State vector and agents' information sets

On date t, the information set of the representative agent is $\Omega_t = \{\underline{x_t}, \underline{y_t}, \underline{w_t}, \underline{c_t}, \underline{N_t}\}$. It includes the current and past observations of all default counts and underlying factors. In the following, the operator $\mathbb{E}_t$ denotes the expectation conditional on the information available at time t, i.e. $\mathbb{E}_t(\bullet) = \mathbb{E}(\bullet|\Omega_t)$.

In the following, we focus on the state vector $X_t = [x_t, y_t, w_t, N_t', N_{t-1}']'$. As shown below, the payoffs of the financial instruments we consider, as well as their values, are functions of X_t (as a result, the information contained in the underlying factor values can be recovered from the observed derivative prices). The tractability of our approach results from the fact that X_t is an affine

Figure 1: Causal scheme



Panel (a) displays the causal scheme underlying our model. Panel (b) represents the scheme prevailing in standard disaster-risk models. Arrows represent Granger-causal relationships.

process. The log conditional Laplace transform, denoted by $\psi(v, X_t)$ and defined by:

$$\mathbb{E}_t\left(\exp(v'X_{t+1})\right) = \exp(\psi(v, X_t)),$$

is affine in X_t. Formally, there exist functions ψ_0 and ψ_1 such that:

$$\psi(v, X_t) = \psi_0(v) + \psi_1(v)'X_t, \tag{5}$$

for the values of v that are such that $\mathbb{E}_t\left(\exp(v'X_{t+1})\right)$ exists. Functions ψ_0 and ψ_1 are made explicit in Appendix A.2 (eqs. a.2 and a.3). As is well-known, the combination of affine processes and an exponential affine stochastic discount factor results in closed-form, or quasi closed-form, expressions for the prices of a wide range of financial instruments (Duffie et al., 2002).

2.5 Preferences, stochastic discount factor and risk-neutral dynamics

The preferences of the representative agent are of the Epstein and Zin (1989) type, with a unit elasticity of intertemporal substitution (EIS).[10] Specifically, the time-t utility of a consumption stream (C_t) is recursively defined by:

$$u_t \quad = \quad (1-\delta)c_t + \frac{\delta}{1-\gamma}\log\left(\mathbb{E}_t \exp\left[(1-\gamma)u_{t+1}\right]\right), \tag{6}$$

where c_t denotes the logarithm of the agent's consumption level C_t, δ the time discount factor and γ the risk aversion parameter.[11] Exploiting the affine property of the state vector X_t, it can be shown that the short-term stochastic discount factor (s.d.f.) at date t, denoted by $M_{t,t+1}$, then admits the following exponential affine representation:

$$M_{t,t+1} \quad = \quad \exp\left[-(\eta_0+\eta_1' X_t) + \pi' X_{t+1} - \psi(\pi, X_t) - \eta_c \varepsilon_t^c - \frac{1}{2}\eta_c^2\right], \tag{7}$$

where scalars η_0 and η_c, and vectors π and η_1 depend on previously-introduced model parameters (see Proposition 2 of Appendix B). Because $\mathbb{E}_t(M_{t,t+1}) = \exp[-(\eta_0+\eta_1' X_t)]$, the short-term risk-free interest rate r_t is affine in X_t and given by:

$$r_t = \eta_0 + \eta_1' X_t. \tag{8}$$

In order to price financial instruments, it is convenient to introduce the risk-neutral probability measure, denoted by $\mathbb{Q}$. This probability measure is defined with respect to the historical one through the change of density $(d\mathbb{Q}/d\mathbb{P})_{t,t+1}$, given by:

$$\frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})} = \exp\left[\pi' X_{t+1} - \psi(\pi, X_t) - \eta_c \varepsilon_t^c - \frac{1}{2}\eta_c^2\right].$$

[10] Using a unit EIS facilitates resolution. Piazzesi and Schneider (2007), or Seo and Wachter (2018), among others, also work under this assumption of a unit EIS. This value is however slightly below the lower bound of the 90% confidence interval found by Schorfheide et al. (2018).

[11] Eq. (6) results from a first-order Taylor expansion around $\rho = 1$ of the general Epstein and Zin (1989) recursive utility defined by: $u_t = \frac{1}{1-\rho}\log\left((1-\delta)C_t^{1-\rho} + \delta\left(\mathbb{E}_t\left[\exp\{(1-\gamma)u_{t+1}\}\right]\right)^{\frac{1-\rho}{1-\gamma}}\right)$, where ρ is the inverse of the EIS.

Appendix B (Proposition 3) shows that X_t is also affine under $\mathbb{Q}$; more precisely it shows that the $\mathbb{Q}$-Laplace transform of X_t is given by:

$$\exp\left(\psi^{\mathbb{Q}}(v,X_t)\right) \equiv \mathbb{E}_t^{\mathbb{Q}}\left(\exp(v'X_{t+1})\right) = \exp\left(\psi_0^{\mathbb{Q}}(v) + \psi_1^{\mathbb{Q}}(v)'X_t\right),$$

with

$$\begin{cases} \psi_0^{\mathbb{Q}}(v) &= \psi_0(v+\pi) - \psi_0(\pi), \\ \psi_1^{\mathbb{Q}}(v) &= \psi_1(v+\pi) - \psi_1(\pi). \end{cases}$$

The fact that X_t is an affine process under $\mathbb{Q}$ facilitates the pricing of various assets whose payoffs depend on future values of X_t. In particular, Appendix C.1 provides closed-form formulas to compute date-t prices of European derivatives with payoffs of the form $\exp(a'X_{t+h})$, $\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}$, $a'X_{t+h}$, or $a'X_{t+h}\mathbb{1}_{\{b'X_{t+h}<y\}}$, settled on date $t+h$. These formulas are key building blocks to price specific financial instruments.

2.6 Pricing credit and equity derivatives

2.6.1 Pricing credit index swaps

A credit index swap allows an investor to either buy or sell protection on a credit index, which is a basket of reference entities. There are two main families of credit indexes, which serve as reference points for CDS markets, that are the Dow Jones CDX and iTraxx indexes. The CDX North American Investment Grade index and the iTraxx Europe main index are each comprised of 125 equally-weighted underlying credits (see Section 3 for more details on the iTraxx Europe main index, which is the one used in our application).

In a credit index swap transaction, a protection seller agrees to pay all default losses in the index in return for a fixed periodic spread $S_{t,h}^{CI}/q$ paid on the total notional of obligors remaining in the index over a period of h years, where q is the number of time periods per year. Should there be no credit event, the protection buyer pays a regular spread until maturity. Upon default of one of the reference entities, the protection seller provides the buyer with the amount that the latter would have lost if she had held the index bond portfolio. For instance, for a \$100,000 position in a

20-name index, with a recovery rate of 50%, the amount would be \$2,500 ($= 50\% \times 100,000/20$). Following this default, the trade continues with the notional amount reduced by the weight of the defaulted credit. In the previous example, the new notional would be \$95,000; the number of reference entities in the index would be reduced to the remaining (non-defaulted) 19 entities.

In our application, we consider that the names in the credit index coincide with segment 1. The payoffs therefore critically depend on $N_{1,t}$. The spread $S^{CI}_{t,h}$ is determined by equalizing the date-t values of the protection leg and of the premium leg, that is:

$$\underbrace{\mathbb{E}^{\mathbb{Q}}_t \left\{ \sum_{k=1}^{qh} \Lambda_{t,t+k}(1-RR)\frac{N_{1,t+k}-N_{1,t+k-1}}{I_1} \right\}}_{\text{Protection leg}} = \underbrace{\frac{S^{CI}_{t,h}}{q}\mathbb{E}^{\mathbb{Q}}_t \left\{ \sum_{k=1}^{qh} \Lambda_{t,t+k}\frac{I_1-N_{1,t+k}}{I_1} \right\}}_{\text{Premium leg}}, \tag{9}$$

where I_1 is the number of entities in segment 1, i.e. the number of names in the index, where RR is the contractual recovery rate, assumed independent of time, and where:

$$\Lambda_{t,t+k} = \exp(-r_t - r_{t+1} - \cdots - r_{t+k-1}), \tag{10}$$

r_t being the risk-free short-term interest rate between periods t and $t+1$.

Hence, credit index swap spreads result from the knowledge of conditional expectations of the form $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k}N_{1,t+k})$ and $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k}N_{1,t+k-1})$, whose computation is addressed in Corollary 1 of Appendix C.1.

Online Appendix O.4 shows that the spread on a CDS written on any entity of segment 1 is also given by eq. (9).

2.6.2 Pricing synthetic Collateralized Debt Obligations (CDOs)

Collateralized Debt Obligations (CDOs), or credit tranches, allow an investor to get a specified exposure to the credit risk of the underlying reference portfolio, or credit index, while in return receiving periodic coupon payments.[12] Losses due to credit events in the underlying portfolio are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritized tranches (junior tranches, mezzanine tranches, followed by senior tranches).

The risk of a tranche is determined by attachment and detachment points. The attachment point, denoted by a, determines the subordination of a tranche. The detachment point, denoted by b, $b > a$, determines the point beyond which the tranche has lost its complete notional. The equity tranche takes the first losses on the portfolio, from $a_1 = 0$ up to b_1. When the portfolio has accumulated losses exceeding the fraction b_1 of notional, the next tranche, (a_2, b_2) with $a_2 = b_1$, will incur losses from any additional defaults up to b_2, and so on.

Let us detail the cash-flows induced by an (a,b) credit tranche in the context of the reference portfolio made of segment-1 entities, that are the iTraxx ones in our study. Consider a protection buyer and a protection seller who meet at date t. Their negotiation results in a spread $S^{TDS}_{t,h}(a,b)$, which is the quote associated with this credit tranche at date t, the maturity date of this derivative product being $t+h$. Let us denote by ℓ_t the cumulative loss, that is:

$$\ell_t = (1 - RR)\frac{N_{1,t}}{I_1}.$$

From dates $t+1$ to $t+h$, cash-flows are exchanged between the two parties unless the cumulative losses ℓ_{t+k} (for $k = 1, \dots, h$) have exceeded the detachment point b. Specifically, at date $t+k$, these cash-flows are the following:

- If cumulative losses ℓ_{t+k} have not reached the attachment point a: (i) there is no cash-flow paid by the protection seller and (ii) the protection buyer pays the full premium $S^{TDS}_{t,h}(a,b)/q$.
- If cumulative losses ℓ_{t+k} exceed the attachment point a, but remain lower than the detachment point b: (i) the protection seller provides the protection buyer with an amount equal to the fraction of the tranche consumed by new losses between $t+k-1$ and $t+k$, that is

[12] The credit-tranche market consists of an actively traded segment and an illiquid "buy-and-hold" segment (Scheicher, 2008). In the actively-traded segment, the underlying credit portfolio is based on the standardized portfolio of a credit index such as the iTraxx or the CDX index. The less-actively-traded segment of the credit-tranche market consists of tailor-made tranches of Collateralized Debt Obligations (CDOs) in which various loans are bundled.

$(\ell_{t+k} - \ell_{t+k-1})/(b-a)$, and (ii) the protection buyer pays a premium equal to the multiplication of the full premium $S^{TDS}_{t,h}(a,b)/q$ by the fraction of the tranche that has not been consumed at date $t+k$, that is $(b-\ell_{t+k})/(b-a)$.

The spread $S^{TDS}_{t,h}(a,b)/q$ is such that the protection and premium legs have the same value at date t, that is:[13]

$$
\begin{aligned}
& \underbrace{\mathbb{E}^{\mathbb{Q}}_t\left\{\sum_{k=1}^{qh}\frac{1}{b-a}\Lambda_{t,t+k}(\min(\ell_{t+k},b)-\max(\ell_{t+k-1},a))\mathbb{1}_{\{a<\ell_{t+k}\}}\mathbb{1}_{\{\ell_{t+k-1}\leq b\}}\right\}}_{\text{Protection leg}} \\
\approx\; & \underbrace{\mathbb{E}^{\mathbb{Q}}_t\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}\frac{\ell_{t+k}-\ell_{t+k-1}}{b-a}\mathbb{1}_{\{a<\ell_{t+k}\leq b\}}\right\}}_{\text{Protection leg}} \\
=\; & \underbrace{U^{TDS}_{t,h}(a,b)+\frac{S^{TDS}_{t,h}(a,b)}{q}\mathbb{E}^{\mathbb{Q}}_t\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}\left(\mathbb{1}_{\{\ell_{t+k}\leq a\}}+\frac{b-\ell_{t+k}}{b-a}\mathbb{1}_{\{a<\ell_{t+k}\leq b\}}\right)\right\}}_{\text{Premium leg}}, \qquad (11)
\end{aligned}
$$

where $U^{TDS}_{t,h}(a,b)$ is an upfront payment and where $\Lambda_{t,t+k}$ is defined in eq. (10).[14] Credit tranches are either quoted in terms of spreads $S^{TDS}_{t,h}(a,b)$, or in terms of up-front payments $U^{TDS}_{t,h}(a,b)$. Typically, in the former case, the up-front payment is fixed, and vice versa.

Appendix C.2 shows that by expanding both sides of eq. (11), computing $S^{TDS}_{t,h}(a,b)$ – or, equivalently, $U^{TDS}_{t,h}(a,b)$ – amounts to calculating date-t prices of payoffs of the forms: $\mathbb{1}_{\{N_{1,t+k}<z\}}$, $N_{1,t+k}\mathbb{1}_{\{N_{1,t+k}<z\}}$, and $N_{1,t+k-1}\mathbb{1}_{\{N_{1,t+k}<z\}}$, these payoffs being settled at date $t+k$. The computation of such prices is addressed in Corollaries 2 and 3 (Appendix C.1).

2.6.3 Pricing equity derivatives

The price P_t of a stock at date t can be deduced from the series of future dividends D_{t+h}, $h \geq 1$, as:

[13] The smaller $\ell_{t+k} - \ell_{t+k-1}$ compared to $b-a$, the better the approximation made on the protection leg. Whereas this approximation overstates the payoff on those dates where $\ell_{t+k-1} < a < \ell_{t+k}$ (because it mistakenly attributes $a - \ell_{t+k-1}$ to the payoff), it understates the payoff when $\ell_{t+k-1} < b < \ell_{t+k}$ (because $b - \ell_{t+k-1}$ is then mistakenly excluded from it).

[14] See e.g. O'Kane and Sen (2003), D'Amato and Gyntelberg (2005), or Morgan Stanley (2011) for an analysis of upfront versus running spread quoting conventions.

$$P_t = \sum_{h=1}^{\infty} \mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+h} D_{t+h}).$$

Let us assume that, as consumption growth, the dividend log growth rate $g_{d,t} = \log(D_t/D_{t-1})$ is affine in $[x_t, y_t, w_t]'$:

$$g_{d,t} = \mu_{d,0} + \mu_{d,x} x_t + \mu_{d,y} y_t + \mu_{d,w} w_t. \tag{12}$$

Proposition 6 (Appendix C.3) provides an approximated solution for the stock returns in the general case where the log growth rate of dividends is affine in X_t. As in Bansal and Yaron (2004), this approximated solution is based on the Campbell and Shiller (1988) linearization of stock returns around the average of the log price-dividend ratio $\tau_t = \log(P_t/D_t)$. In the solution, τ_t is expressed as an affine function of X_t.

The payoffs of equity derivatives depend on P_t. The dynamics of P_t is deduced from the dynamics of the ex-dividend return $r^*_{t+1} = \log(P_{t+1}/P_t)$. This return is given by:

$$r^*_{t+1} \quad = \quad \log\left(\frac{P_{t+1}}{D_{t+1}} \times \frac{D_t}{P_t} \times \frac{D_{t+1}}{D_t}\right) = \tau_{t+1} - \tau_t + g_{d,t+1}, \tag{13}$$

which is affine in $[X'_{t+1}, X'_t]'$. We therefore have, for any horizon h:

$$P_{t+h} \quad = \quad P_t \exp\left(r^*_{t+1} + \cdots + r^*_{t+h}\right) \tag{14}$$

$$\quad = \quad P_t \exp\left(\tau_{t+h} - \tau_t + g_{d,t+1} + g_{d,t+2} + \cdots + g_{d,t+h}\right). \tag{15}$$

Let us consider the price of a European put option of maturity h and strike K. This price is given by $\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}(K - P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right)$. Using eq. (14), we obtain:

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}(K - P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right) \\ = \; & K\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\mathbb{1}_{\{r^*_{t+1}+\cdots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\ & -P_t\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp(r^*_{t+1} + \cdots + r^*_{t+h})\mathbb{1}_{\{r^*_{t+1}+\cdots+r^*_{t+h}<\log(K)-\log P_t\}}\right). \end{aligned} \tag{16}$$

Appendix C.4 provides details about the computation of the two conditional expectations appearing on the right-hand side of eq. (16).

3 Data

The data cover both financial and macroeconomic series spanning the period from January 2006 to September 2017, at a bi-monthly frequency. On the financial side, we use credit index swap spreads and prices of tranches associated with the iTraxx Europe main index. Furthermore, the financial data include prices of far-out-of-the-money (far-OTM) equity put options written on the EURO STOXX 50, which is one of the most important benchmarks of European equity markets. On the macroeconomic front, data consist of quarterly real private consumption, taken from the Area-Wide Model (AWM) database (Fagan et al., 2001). A cubic spline is applied to the quarterly series in order to obtain one at a bi-monthly frequency.[15]

In what follows, we detail the financial data and provide, in particular, arguments for the systemic nature of iTraxx constituents (Subsection 3.2) and their stability across time (Subsection 3.3).

3.1 Financial data description

3.1.1 Credit index and tranche prices (iTraxx)

To estimate the model, we employ data based on the iTraxx Europe main index, a euro-denominated index involving 125 large European firms whose credit default swaps are actively traded. iTraxx indexes roll every six month – forming series. That is, every six months, a new series of the index is created with updated constituents. Derivatives written on previous series continue trading, although liquidity is concentrated on options written on the on-the-run series (see Markit, 2014).

The roll consists of a series of steps which are administered by Markit, a financial services information company that owns and compiles CDX and iTraxx indexes. For the Markit iTraxx Europe indexes, liquidity lists are formed from the trading volumes from the Depository Trust and Clearing Corporation (DTCC) Trade Information Warehouse.[16] Markit then applies index rules

[15] As shown on Panel (b) of Figure 4 (compare the black solid line and the gray dotted line), this smoothing only removes a fairly small part of the variance of consumption growth.

[16] http://www.dtcc.com/derivatives-services/trade-information-warehouse.

to determine the index constituents among the most liquid names (see Markit, 2016). For iTraxx Europe main (the index used in this study), the final Index comprises 30 Autos & Industrials, 30 Consumers, 20 Energy, 20 Telecommunications, Media and Technology (TMTs) and 25 Financials.

Constituents of the iTraxx Europe main index must have an investment grade rating. That is, to be included in the list of constituents, entities have to be rated BBB-/Baa3/BBB- (Fitch/-Moody's/S&P), or higher. On average, over the ongoing life of the iTraxx index (from series 1 to 30), the median rating of its constituents is BBB+ at the S&P rating – which corresponds to a Baa1 Moodys' rating.

We extract spreads of iTraxx indexes from Thomson Datastream. These spreads correspond to maturities of 3, 5, 7 and 10 years. We also use iTraxx tranche prices that come from Markit's website.[17] For each maturity, we use prices associated with the following tranches: 0%-3%, 3%-6%, 6%-9%, 9%-12% and 12%-22%. We do not use prices associated with the super-senior tranche (22%-100%) as well as prices associated with the 10-year maturity given the very low liquidity of these contracts. Note also that, for liquidity reasons, our Markit data do not cover all dates in our sample. In particular, we do not have tranche prices before January 2007 and after March 2013.

Because each index roll features fixed maturity dates, market prices are not of the "constant-maturity" type. To deal with this issue, for each considered maturity, for (i) each date and (ii) each pair of attachment/detachment points, we look for the tranche price whose residual maturity is the closest to the considered one. If the residual maturity of the resulting tranche is not in a ± 1 year window around the targeted maturity, no price is reported.

3.1.2 Equity options (EURO STOXX 50)

Equity put options are far out-of-the-money options written on the EURO STOXX 50 index. We consider two maturities, 6 and 12 months, and strikes equal to 70% of the current value of the index. These options protect against larger-than-30% falls in the equity index. That is, the payoffs of these

[17] http://www.creditfixings.com/CreditEventAuctions/itraxx.jsp. For each date, maturity and tranche, we convert all quotes into an equivalent running spread with no upfront payment by using the risky duration approach (see e.g. O'Kane and Sen, 2003; D'Amato and Gyntelberg, 2005; Morgan Stanley, 2011).

options become strictly positive in case of a fall of the index by more than 30%. Such option prices are not directly available on Thomson Datastream; option prices reported on those database are for contracts with standardized maturity dates and strikes. We compute the prices of our out-of-money options by applying interpolation splines on available data, in both the time and strike (1000, 1500, 2000, 2500, 3000, 3500, 4000 euros) dimensions. Following market convention, we convert put option prices into implied volatilities using the Black-Scholes formula and Euribor swap rates as the risk-free rate.

3.2 The systemic nature of iTraxx entities

In this subsection, we document that iTraxx constituents represent substantial shares of European economies – along different dimensions. This supports the idea that these entities are large enough for their defaults to have economy-wide effects, which supports our use of the iTraxx index to proxy for systemic firms.

Table 1 reports information, collected on Thomson Reuters Eikon, on the 125 entities of series 30 of the iTraxx Europe main index (series 30 was issued on September 20, 2018). Specifically, the Table reports the different countries whose firms are included in the iTraxx's series 30, their number of iTraxx firms (as of September 20, 2018) and their respective market capitalization, number of employees, long-term debt and total debt, as a percentage of the total listed firms in a given country. On average, across the fifteen different countries, iTraxx entities represent about 27% of the market capitalization of a country; with the Netherlands featuring the highest proportion of total market capitalization amounting to 62%. By the same token, the average number of employees of iTraxx entities represents about 22% of the total number of employees hired by the listed firms of a given country; with Germany having the largest proportion (44%). Last, iTraxx entities represent, on average, about 42% of the long-term debt and total debt of all the listed firms of a country.

Aggregating these metrics across all 125 entities amounts to about 5 trillion euros of market capitalization, 12.5 million employees, 3.8 trillion euros of long-term debt and 5.5 trillion euros of total debt.[18] These descriptive statistics suggest that iTraxx entities are very large and may

[18]To provide a sense of the order of magnitude of these figures, note that the aggregate size of the market capital-

Table 1: Systemic nature of iTraxx entities

Country	Nb. iTraxx entities	Market capitalization	Nb. employees	Long-term debt	Total debt
Austria	1	3.88	3.44	3.45	3.27
Belgium	2	45.23	36.85	44.31	38.41
Denmark	1	3.64	2.29	65.19	70.08
Finland	1	3.46	1.37	3.00	2.36
France	29	50.25	41.78	71.64	64.48
Germany	21	41.10	43.70	65.29	69.27
Italy	7	40.55	31.08	61.16	60.08
Luxembourg	2	11.56	27.26	13.29	13.93
Netherlands	11	62.14	41.04	77.07	74.63
Norway	2	31.71	10.98	4.72	5.39
Portugal	1	23.61	–	–	–
Spain	6	8.07	26.73	68.43	64.76
Sweden	3	8.50	9.54	4.70	5.15
Switzerland	7	29.23	30.92	56.85	62.94
United Kingdom	31	37.43	27.63	51.22	55.06

This table reports the different countries whose firms are included in the iTraxx's series 30, their number of iTraxx firms (as of September 20, 2018) and their respective market capitalization, number of employees, long-term debt and total debt, as a percentage of the total listed firms in a given country. All figures are in percentages. The data on iTraxx firms and on all listed firms per country is collected from Thomson Reuters Eikon. As of the time of the analysis, Portugal's only iTraxx firm (EDP Finance) does not disclose its number of employees, long-term debt and total debt.

have a direct impact on the health of other firms, via their financial, legal or business relationships (echoing the arguments of Azizpour et al., 2018). Indeed, contagion is not limited to the financial sector. This is supported by the fact that General Motors and Chrysler received 20% of the funds of the Troubled Asset Relief Program (launched in 2008), amounting to about 80 billion dollars. The arguments used at the time were that millions of jobs would be lost and that their default could trigger the closure of many factories, the liquidation of suppliers and dealerships and the potential loss of an entire industry.[19]

3.3 The stability of iTraxx indexes

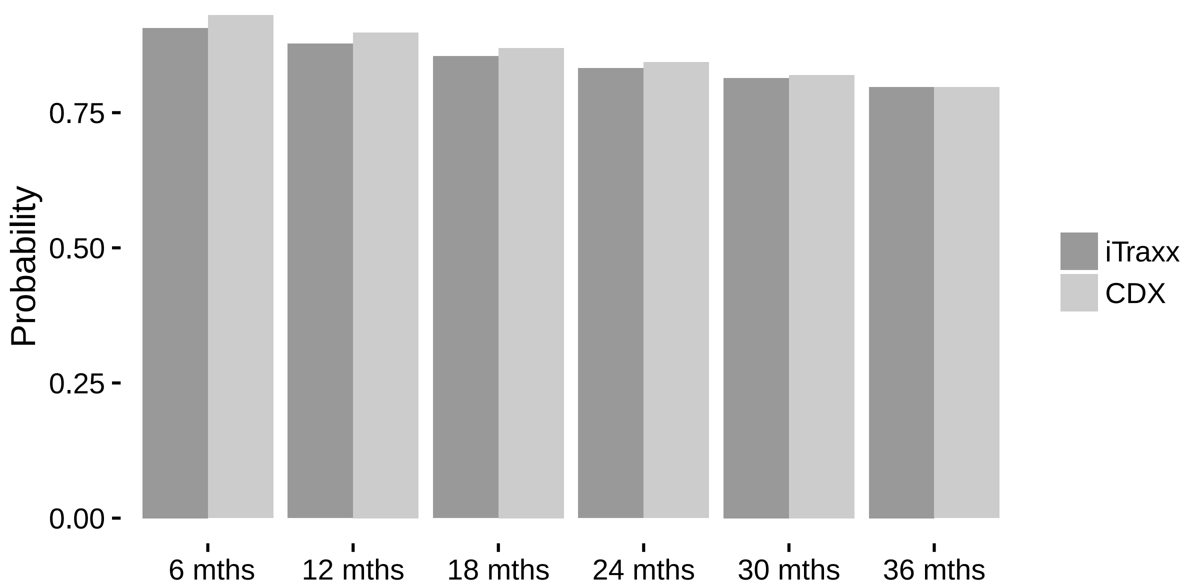
As mentioned above, iTraxx indexes roll every six months. In this subsection, we want to see whether the rolling of the index affects the stability of its constituents across time.

ization is about twice as large as the French GDP in 2017, while the aggregate number of employees represents more than half of the active population of Spain.

[19]See e.g. https://www.reuters.com/article/autos-bailout-study-idUSL1N0JO0XU20131209.

Figure 2 reports statistics on the stability of the iTraxx across its entire past history, up to series 30. For comparison, we perform similar statistics on the CDX North American Investment Grade index (up to series 31). The j^{th} bar depicts the average proportion of constituents that belong to a given credit default swap index (iTraxx or CDX) series and the one prevailing j semesters later. For instance, the first (respectively second) bar is obtained by computing the proportion of iTraxx constituents that belong to the index at 6 months intervals (respectively 12 months intervals).

Figure 2: iTraxx constituents' stability



This figure illustrates the stability of the constituents of the iTraxx index (dark gray bars) and the CDX index (light gray bars). The j^{th} bar depicts the average proportion of constituents that belong to a given credit default swap index (iTraxx or CDX) series and the one prevailing j semesters later. For instance, the first (respectively second) bar is obtained by computing the proportion of iTraxx constituents that belong to the index at 6 months intervals (respectively 12 months intervals). The computations are based on the 30 first iTraxx Europe main index series and on the 31 first CDX North American Investment Grade series, respectively.

We find that, on average, over the first 30 iTraxx series spanning the last 15 years, about 80% of the constituents are the same in a given series and the one that is launched three years later. As shown by the low turnover in Figure 2, iTraxx constituents are fairly stable from one series to another. Therefore, in spite of the fact that our estimation sample covers several iTraxx indexes, we consider a single model parametrization (see Section 4).[20]

[20] By contrast, Collin-Dufresne et al. (2012) modify their model calibration for each new CDX index (see their Table 1). This table discloses important changes in the parameter values they obtain from one index to another, which is at odds with the stability of the index constituents (see Figure 2).

Given the constituents are relatively stable across series, conducting a similar analysis to that of Subsection 3.2 on any other series would lead to qualitatively similar results.

4 Estimation

Bringing the model to the data amounts to determining two types of objects: the model parameters have to be estimated and the latent variables have to be filtered. In order to discipline the estimation, some of the model parameters – in particular the preference parameters – are not estimated, but are calibrated. Thanks to the tractability of our framework, the estimation of remaining parameters and the filtering of unobserved variables are performed jointly by Kalman filter techniques. This estimation approach could not be employed in frameworks that do not entail closed-form formulas.[21]

4.1 Calibrated parameters

The left panel of Table 2 reports the calibrated parameters. Following Seo and Wachter (2018), the risk aversion parameter γ is set to 3 and the annualized rate of time preference to 1.2%. Because our model is at a bi-monthly frequency, this rate of time preference translates into $\delta = (1-1.2\%)^{1/6} \approx 0.998$. As mentioned above (Subsection 2.5), we consider a unit elasticity of intertemporal substitution. Another calibrated moment is the population expectation of consumption growth, that is set to 1.5% (annualized). The variance of the annual consumption growth rate is set to 5%, which is roughly consistent with the values given by Barro and Ursua (2011) for the OECD.[22] The standard deviation of the volatile component of consumption growth, that is σ_c, is set to 0.8%, which implies that a relatively small share – about 15% – of the annual consumption variance is driven

[21] On a standard laptop, the computation of the likelihood function – which involves the estimation of the latent factors – takes about ten seconds. Though this is not a negligible amount of time, it still allows for the numerical optimization of the likelihood function with respect to a reasonable number of parameters (about ten here). By contrast, Seo and Wachter (2018) have to employ a 200-cores High Performance Computer (HPC) cluster to evaluate CDX prices (on a single set of calibrated parameters); this suggests that any form of numerical optimization would be particularly demanding in the context of their model.

[22] According to Barro and Ursua (2011, Table 2), the standard deviation of the annual consumption growth rate of OECD countries has been of 5.7% for a large sample starting at the end of the 19th century and ending in 2009 – and of 2.9% for a post-world-war-II sample.

by this volatile component. As in Bansal and Yaron (2004), the log growth rate of real dividends is given the same marginal expectation as the log growth rate of consumption (1.5%, annualized). We take a contractual recovery rate RR of 40%, consistently with standard market practice. We also set the average default rate of the systemic entities to be of 0.3% per year. This is consistent with historical data on investment-grade entities compiled by Moody's.[23]

4.2 State-space model

During the period we consider (2006-2017), there has been no systemic default in the euro area. On October 22 2009, though, CDS contracts written on the French electronics firm Thomson – one of the iTraxx constituents – were triggered. However, we do not consider this credit event to be a systemic event. Indeed, this credit event was not a failure of the firm, but a restructuring of its debt.[24] In the U.S., following the so-called "Big Bang" changes in practices on credit events (April 8 2009) restructuring was excluded from the list of credit events triggering American CDSs (see Coudert and Gex, 2010).

Accordingly, we have $n_t^s = 0$ and therefore $w_t = 0$ for all dates t in our sample. Then we can focus on the filtering of the other factors x_t and y_t. Let us stress that, in spite of the fact that $w_t = 0$ over our sample, the threat of possibly having $w_{t+k} > 0$, $k > 0$, is taken into account by investors on each date t of the sample. Accordingly, the parameters governing the dynamics of w_t, i.e. ξ_w, μ_w, $\mu_{c,w}$ and $\mu_{d,w}$ are, in particular, identifiable through observed derivative prices.

Observed variables include credit index swap spreads of different maturities, tranche spreads and equity put prices. Let us denote by Γ_t the vector of observed data – that includes financial data as well as consumption growth – and by θ the vector of model parameters to be estimated. Over our estimation period, our model predicts that these prices are functions of $z_t = [x_t, y_t]'$ (and of $w_t = 0$) and θ. Allowing for measurement errors denoted by ε_t, the set of measurement equations

[23] More precisely, this corresponds to the average cumulative issuer-weighted global default rates for Baa-rated firms on the period 1920-2016 (see Moody's, 2017, Exhibit 32). On average, over the ongoing life of the iTraxx index (from series 1 to 30), the median rating of its constituents is BBB+ at the S&P rating – which corresponds to a Baa1 Moodys' rating.

[24] The recovery rate was determined through auctions; for the shortest maturity (2.5 years), the recovery rate was of 96.26%.

reads:

$$\Gamma_t = F(z_t;\theta) + \varepsilon_t, \tag{17}$$

where the components of ε_t are mutually and serially independent Gaussian shocks, i.e. $\varepsilon_t \sim i.i.d.\mathcal{N}(0,\Sigma_\varepsilon)$, where Σ_ε is a diagonal matrix.

The transition equation describes the dynamics of z_t. Using the formula provided in Appendix A.1, the dynamics of z_t can be expressed as follows:

$$z_{t+1} = \mu_z + \Phi_z z_t + \Sigma_z^{1/2}(z_t)\xi_{t+1}, \tag{18}$$

where ξ_{t+1} is a martingale difference sequence that, conditional on Ω_t, is zero mean and admits an identity covariance matrix. Thus, matrix $\Sigma_z(z_t)$ describes the conditional heteroscedasticity.

Eqs. (17) and (18) constitute the state-space form of our model. We employ the extended Kalman filter to approximate the log-likelihood function associated with this state-space model.[25] By maximizing this function with respect to θ, we obtain estimates of the parameters that have not been directly calibrated (Subsection 4.1) or that cannot be retrieved from calibrated moments.[26] A final pass of the Kalman algorithm provides us with filtered values of the latent factors z_t.

5 Results

5.1 Model fit

Table 2 shows calibrated and estimated parameters. It notably appears that c_j parameters ($j \in \{1,2\}$) are equal to 0.38, revealing a substantial level of contagion. It implies that an additional default by one systemic firm on date t leads to an increase in the expected number of systemic

[25]Derivatives of function F with respect to z_t are obtained numerically. In order to reduce the number of parameters to estimate, the diagonal entries of Σ_ε (variances of the measurement errors) are calibrated in a preliminary step. We employ the approach of Green and Silverman (1994) and proceed as follows: we apply a smoothing spline to series of observed prices. Next, we compute the sample variances of the differences between the prices and their smoothed counterparts. The variances of the measurement equations are set to these values.

[26]In our framework, this approach notably benefits from the existence of closed-form formulas to compute calibrated moments (these computations are based on the results shown in Online Appendix O.2).

default on date $t+1$ by 0.76 (2×0.38) on date $t+1$.[27] Responses to systemic defaults will be studied more extensively through impulse response functions in Subsection 5.2. Following Abel (1999), Collin-Dufresne et al. (2016) and Seo and Wachter (2018), we assume that, up to an affine transformation – and up to consumption-specific Gaussian disturbances (ε_t^c in eq. 4) – the log growth rate of dividends is equal to consumption growth. That is, the parameters $\mu_{d,x}$, $\mu_{d,y}$ and $\mu_{d,w}$ pertaining to eq. (12) are proportional to the parameters $\mu_{c,x}$, $\mu_{c,y}$ and $\mu_{c,w}$ pertaining to eq. (4). In our setup, this "leverage parameter" χ is equal to 2.5, which is in line with the values found in Abel (1999), Collin-Dufresne et al. (2016) and Seo and Wachter (2018) (2.74, 2.5 and 2.6, respectively). Moreover, the fact that $\rho_x = 0.988$ and $\rho_y = 0.829$, with associated half-lives of 9.6 and 0.6 years, respectively, indicates that the persistence of x_t is larger than that of $y_t - x_t$. This is illustrated by Figure 3, which displays the filtered factors x_t and y_t. This figure shows that y_t is more volatile than x_t. In particular, contrary to x_t, process y_t has been more sensitive to the post-Lehman crisis (late 2008, early 2009) and to the peak of the euro-area sovereign debt crisis (late 2011, early 2012).

Figure 3 also shows that the long-run factor x_t remained subdued before the euro-area sovereign debt crisis of 2010-2012. Therefore, the peak reached by y_t in late 2008 is mainly due to an increase in the shorter-run component of y_t, i.e. $y_t - x_t$. This suggests that, as regards corporate European credit risk, the post-Lehman crisis was then perceived as a relatively short-lived phenomenon. By contrast, the European sovereign debt crisis triggered an increase in the long-run component of y_t. This indicates that the market considers that this latter crisis will have a longer-lived impact on corporate European credit risk. As of the end of the sample, the level of the long-run factor x_t is higher than before 2008. As a consequence, even though the 2017 level of y_t is below its late 2008 level, conditional medium to long-run expectations of y_t are higher in 2017 than by in late 2008.

Panel (a) of Table 3 reports model-implied population moments. It indicates for instance that the average excess return for our stock index is of 5.7% and that the maximum Sharpe ratio, evaluated at the average values of the state vector X_t, has a value of 75%, which is for instance

[27] Eq. (2) implies that $\mathbb{E}_t(n_{j,t}|\underline{x_t}, \underline{y_t}, \underline{n^s_{t-1}}) = \beta_j y_t + c_j n^s_{t-1}$ and therefore that $\mathbb{E}_t\left(n_{1,t} + n_{2,t}|\underline{x_t}, \underline{y_t}, n^s_{t-1} = K+1\right) - \mathbb{E}_t\left(n_{1,t} + n_{2,t}|\underline{x_t}, \underline{y_t}, n^s_{t-1} = K\right) = 2c_j$.

comparable to the 70% maximum Sharpe ratio value reported in Brennan et al. (2004).[28] Panels (b), (c) and (d) of Table 3 documents the fit resulting from our estimation approach by comparing the sample averages of observed financial data to their model-implied counterparts. Beyond our baseline estimates (reported on the second column), the table also includes the estimates of two counterfactual exercises: (i) without contagion (i.e. setting $c_1 = c_2 = 0$ and modifying $\beta_1 = \beta_2$ such that the average default probability of systemic entities is the same as in the baseline model) and (ii) without macroeconomic effects (i.e. setting $\mu_{c,w} = 0$). By imposing no contagion, we prevent the formation of default clusters while maintaining the same probability of default we had under our baseline estimation. We thus observe that, on the one hand, spreads for senior tranches practically vanish, which reflects the fact that there are fewer default clusters and hence a lower probability of these tranches being triggered. On the other hand, spreads for equity tranches increase because now defaults are evenly spread in time and therefore these tranches are more likely to be triggered. Moreover, by switching off macroeconomic effects, the default of a systemic entity no longer has an effect on consumption – *conditional on x_t and y_t*. By eliminating the possibility of a dramatic drop in consumption due to a systemic default, consumption becomes less volatile (the standard deviation of consumption growth goes from 5% in the baseline case to 2%, see second row of Table 3) and a systemic default no longer needs to occur *only* in a bad state of the economy; this implies that investors no longer require risk premiums that are as high as under our baseline estimation. Consequently, we observe that the average equity excess return and maximum Sharpe ratio are muted and all spreads (indexes and tranches) decrease substantially. The counterfactual exercises support the conjecture that both channels – contagion and macroeconomic effect – are important for the model fit.

The model fit is also illustrated by Figures 4 to 8. Figure 4 depicts the fit of stock returns and consumption growth. In our framework, model-implied stock returns are given by a linear combination of the state vector X_t and and its lag X_{t-1} (see eq. a.12). Let us stress that our estimation approach does not directly involve stock returns; in this sense, the fit shown on Panel (a) is "out-of-sample". Panel (b) of Figure 4 demonstrates that the model-implied persistent component

[28]The importance of Sharpe ratios to match empirical regularities across markets is highlighted by Chen et al. (2009). Appendix O.5 details the computation of the maximum Sharpe ratio in our context.

of Δc_t (i.e. $\mu_{c,0} + \mu_{c,x} x_t + \mu_{c,y} y_t$, see eq. 4) captures an important share of observed consumption growth fluctuations. Though the consumption process also underlies the pricing of credit derivatives in Christoffersen et al. (2017) or in Seo and Wachter (2018), the latter two papers do not discuss the ability of their model to track consumption's dynamics. Figure 5 complements the information provided on Figure 4 by displaying the unconditional distribution of year-on-year consumption growth. Panel (a) of this figure shows the probability distribution function (p.d.f.) of this distribution and Panel (b) shows its cumulative distribution function (c.d.f.).[29] The fact that this distribution features a fat left tail is better seen on Panel (b), which shows in particular that while the probability of observing a drop in consumption by more than 5% is close to 2% in our baseline model, it would be (virtually) zero in the absence of disastrous events ($\mu_{c,w} = 0$, case represented by the dashed line).

Figure 6 illustrates the fit of the iTraxx index swap spreads of different maturities. Figure 7 compares observed and model-based implied volatilities of far-OTM put options and Figure 8 displays tranche price estimates. These figures show that the model is successful in capturing the main joint fluctuations of consumption growth, stock returns and stock and credit derivatives exposed to systemic risk. In particular, our model is able to fit stock returns (computed on the EURO STOXX 50) despite the fact that this series is not included in the estimation. Moreover, although we use a longer sample (2006-2017 versus 2005-2008) –including acute crisis periods– and a larger cross-section of prices than in Collin-Dufresne et al. (2012), Christoffersen et al. (2017), or Seo and Wachter (2018), the fit we obtain for credit derivatives is comparable to theirs.[30]

[29] Closed-form formulas for the conditional c.d.f. of a linear combination of the state vector X_t can be deduced from a straightforward adaptation of Corollary 2 (see Appendix C.1.1). At a bi-monthly frequency, annual consumption growth is an affine combination of six consecutive values of the state vectors X_t (see eq. 4).

[30] Unlike our analysis (which uses the iTraxx Europe main index), these studies focus on (U.S.) CDX-index data. However, the 10-year index swap spreads of both CDX and iTraxx indexes share similar dynamics and descriptive statistics. Specifically, the 10-year spreads on the iTraxx index feature a slightly lower mean (of about 98 basis points, relative to its CDX-counterpart of 107 basis points) and higher standard deviation (of about 40 basis points, relative to its CDX-counterpart of 30 basis points). Moreover, regressing 10-year iTraxx index swap spreads on 10-year CDX index swap spreads (respectively, regressing the squared values of 10-year iTraxx index swap spreads on the squared values of 10-year CDX index swap spreads) we find that, at a 1% significance level, we fail to reject that the intercept is statistically different from zero and that the slope coefficient is statistically different from 1, while the reported R-squared is of 75% (resp. 61%).

Table 2: Estimated parameters

Panel (a) – Calibrated parameters			Panel (b) – Estimated parameters			
γ		3	$c_i \quad i \in \{1,2\}$		0.38	[0.00]
δ		0.997				
EIS		1	$\beta_i \quad i \in \{1,2\}$	$(\times 10^2)$	1.42	[0.01]
			μ_w	$(\times 10^{-2})$	3.11	[0.68]
$\mathbb{E}(\Delta c_t)$	$(\times 6)$	1.50%	ξ_w	$(\times 10^2)$	5.14	[1.14]
$s.d.(\Delta c_t + \cdots + \Delta c_{t-5})$		5.00%				
σ_c		0.80%	μ_x	$(\times 10^2)$	0.81	[0.27]
$\mathbb{E}(g_{d,t})$	$(\times 6)$	1.50%	μ_y	$(\times 10^2)$	6.19	[1.78]
			ρ_x		0.988	[0.00]
			ρ_y		0.831	[0.02]
			$\mu_{c,x}$	$(\times 10^4)$	-3.06	[0.90]
			$\mu_{c,y}$	$(\times 10^4)$	-6.74	[1.49]
			$\mu_{c,w}$	$(\times 10^4)$	-4.18	[0.31]
			$\mu_{d,x}$	$(\times 10^4)$	-7.91	[3.66]
			$\mu_{d,y}$	$(\times 10^4)$	-17.40	[7.11]
			$\mu_{d,w}$	$(\times 10^4)$	-10.80	[2.11]

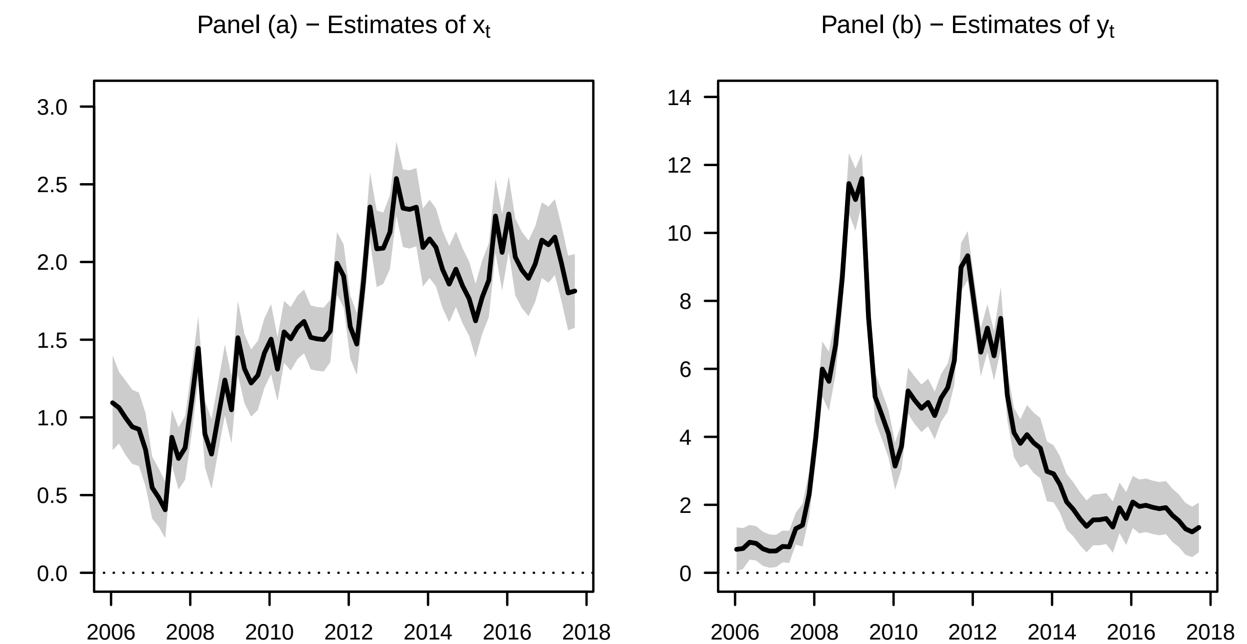
This table presents the model parameterization. γ is the coefficient of relative risk aversion, δ is the time discount factor. EIS is the elasticity of intertemporal substitution. $\mathbb{E}(\Delta c_t)$ and $\mathbb{E}(g_{d,t})$ are multiplied by 6 so as to be expressed in annualized terms. The parameterization is such that $\mathbb{E}(x_t) = \mathbb{E}(y_t) = 1$ (see Appendix A.1). Parameters c_i and β_i define the conditional distribution of the number of defaults in segment i on date t, given in eq. (2). Parameters ξ_w and μ_w define the conditional distribution of w_t, which is given in eq. (3). Parameters μ_x, μ_y, ρ_x and ρ_y characterize the dynamics of factors x_t and y_t (see eq. 1). The specification of the consumption growth rate is given by eq. (4), that is: $\Delta c_t = \mu_{c,0} + \mu_{c,x} x_t + \mu_{c,y} y_t + \mu_{c,w} w_t + \sigma_c \varepsilon_t^c$. $s.d.(\Delta c_t + \cdots + \Delta c_{t-5})$ denotes the unconditional standard deviation of consumption growth. The specification of the dividend growth rate is given by eq. (12), that is $g_{d,t} = \mu_{d,0} + \mu_{d,x} x_t + \mu_{d,y} y_t + \mu_{d,w} w_t$. It is assumed that $[\mu_{d,x}, \mu_{d,y}, \mu_{d,w}] = \chi[\mu_{c,x}, \mu_{c,y}, \mu_{c,w}]$. Panel (a) reports calibrated parameters. Panel (b) reports parameters estimated by maximizing an approximation of the log-likelihood associated with the state-space model defined by measurement equations (17) and transition equations (18) (see Subsection 4.2). Standard deviations (in square brackets) are calculated from the outer product of the log-likelihood gradient, evaluated at the estimated parameter values.

Table 3: Model fit and counterfactuals

	Data	(I) Baseline	(II) $c = 0$	(III) $\mu_{c,w} = 0$
Panel (a) Model-implied population moments (in percent)				
Avg. annual consumption growth	2.00^a	1.49	1.49	1.49
St. dev. annual consumption growth	1.5^a / 2.9^b / 5.7^c	5.00	4.34	2.07
Avg. short-term risk-free rate	1.48^a	2.09	2.09	2.70
St. dev. short-term risk-free rate	2.54^a	0.86	0.58	0.08
Avg. equity excess return		5.69	4.37	1.07
Maximum Sharpe ratio		74.9	34.9	4.7
Avg. default proba. of a systemic entity	0.30^d	0.30	0.30	0.30
Panel (b) ITRAXX indices (in b.p.)				
3 years	65	60	40	33
5 years	88	72	37	31
7 years	101	84	35	30
10 years	112	115	35	30
Panel (c) ITRAXX tranches (in b.p.)				
3 years, Tranche: 0-3%	1879	1580	2488	1218
3 years, Tranche: 3-6%	772	543	79	269
3 years, Tranche: 6-9%	452	321	7	111
3 years, Tranche: 9-12%	160	230	6	47
3 years, Tranche: 12-22%	113	80	0	5
5 years, Tranche: 0-3%	1444	1271	2063	957
5 years, Tranche: 3-6%	663	481	142	238
5 years, Tranche: 6-9%	421	295	10	105
5 years, Tranche: 9-12%	151	235	4	52
5 years, Tranche: 12-22%	91	108	0	8
7 years, Tranche: 0-3%	1241	1172	1917	856
7 years, Tranche: 3-6%	672	470	206	231
7 years, Tranche: 6-9%	439	290	19	104
7 years, Tranche: 9-12%	146	231	3	52
7 years, Tranche: 12-22%	94	113	0	9
Panel (d) Implied Volatility (in p.p.)				
Maturity: 6 months	33%	30%	36%	13%
Maturity: 12 months	30%	31%	33%	10%

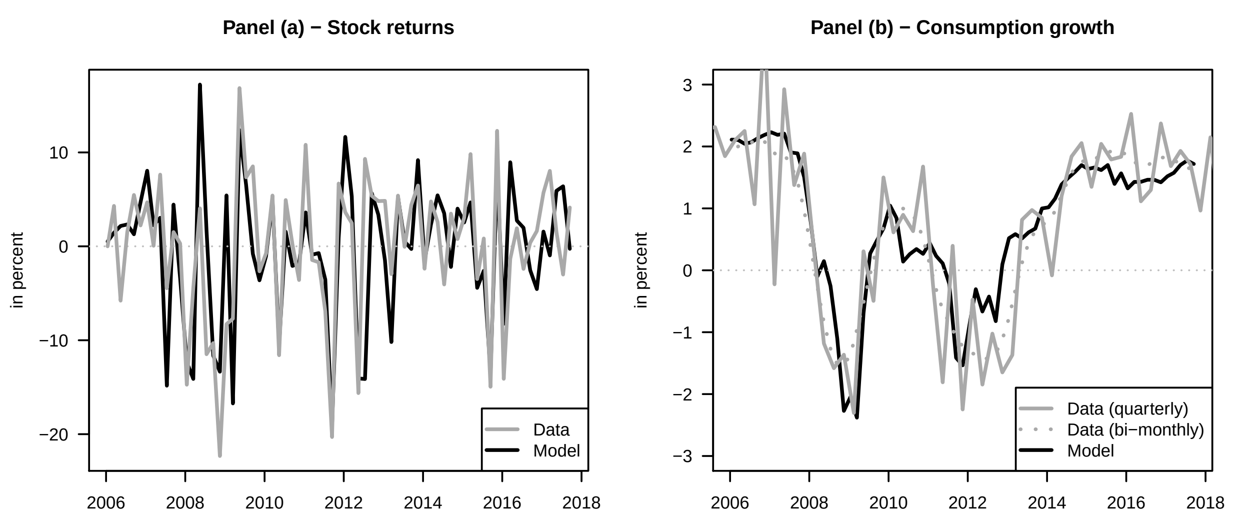
[a]: These moments are based on the Area-wide-Model (AWM) database for the euro area (Fagan et al., 2001, updated database covering the period from 1970Q1 to 2017Q4); [b] and [c] come from Barro and Ursua (2011, Table 2, OECD countries), [b] (respectively [c]) is based on a large sample starting at the end of the 19th century and ending in 2009 (resp. a post-world-war-II sample); [d] is based on Moody's (2017) (see Footnote 23). The reported maximum Sharpe ratio is evaluated at the population mean of the state vector, i.e. for $X_t = \bar{X}$ (a one-year investment is considered here; see Online Appendix O.5 for computational details). Panels (b), (c) and (d) report sample averages. Model-implied prices are evaluated by setting factors x_t and y_t to their filtered values derived from the extended Kalman filter (see Subsection 4.2). Models (II) and (III) are modified versions of Model (I), which is the baseline model; in Model (II), there is no contagion (i.e. $c_1 = c_2 = 0$) and $\beta_1 = \beta_2$ are modified so as to keep the same average default probability of systemic entities; in Model (III), defaults do not affect consumption (i.e. $\mu_{c,w} = 0$; see eq. 4).

Figure 3: Estimated factors x_t and y_t



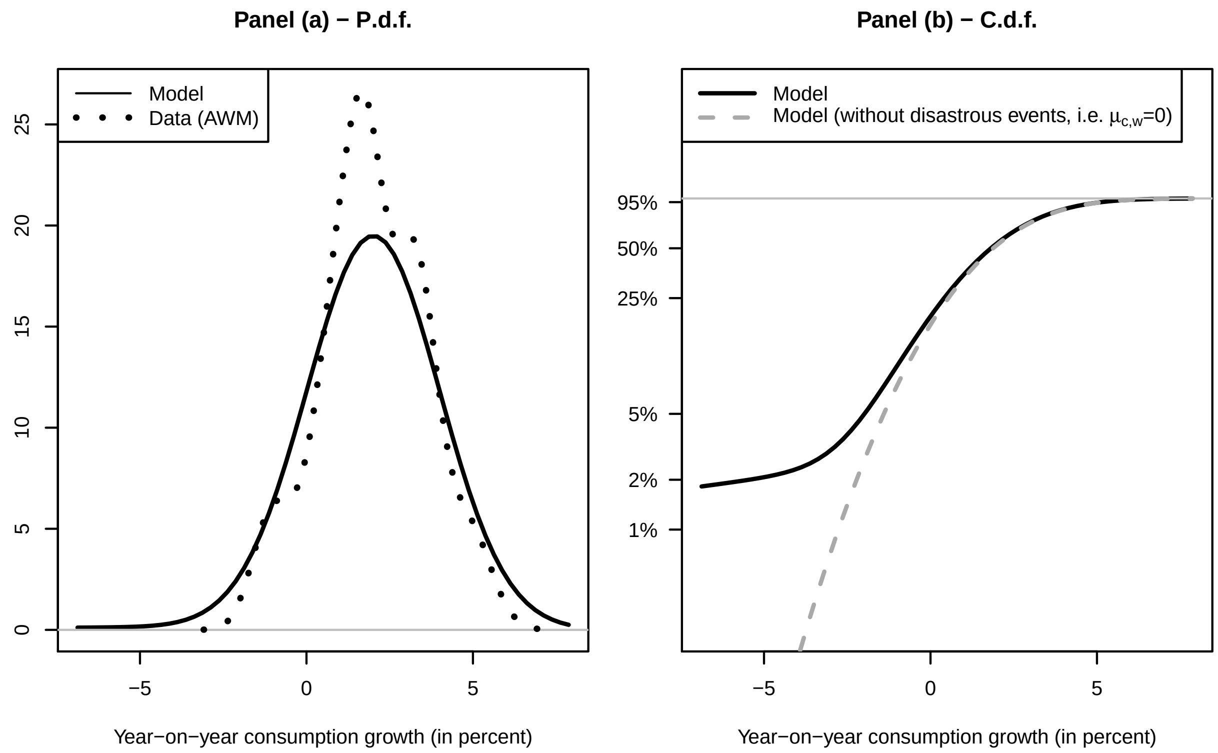
This figure displays the filtered values of x_t and y_t. These values stem from the extended Kalman filter applied on the state-space model whose measurement and transition are eqs. (17) and (18), respectively. Gray-shaded areas are 95% (prediction) confidence intervals.

Figure 4: Fit of stock returns and consumption growth



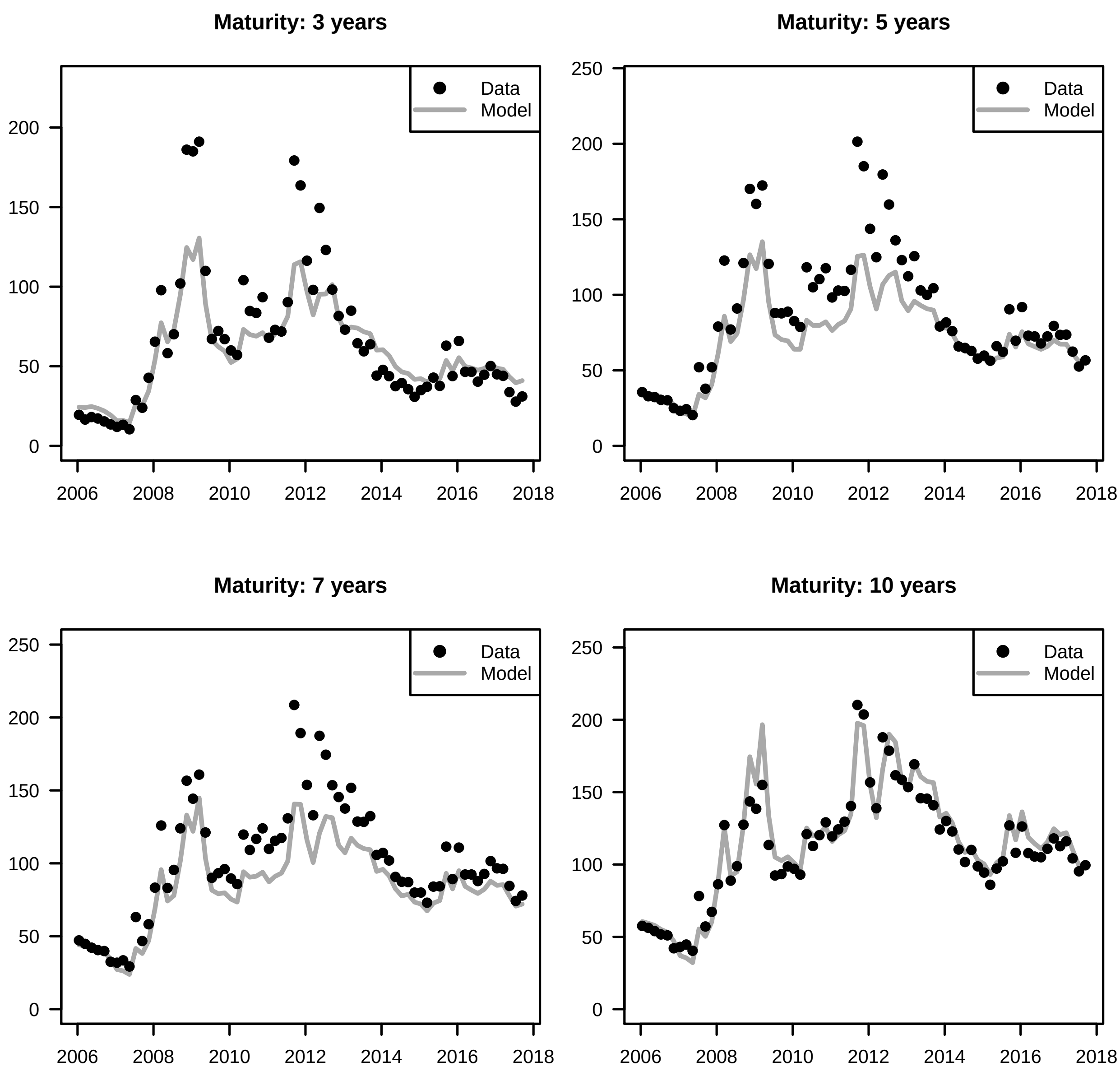
This figure compares model-implied and observed series of stock returns and consumption growth. Stock returns (Panel (a)) are computed on the EURO STOXX 50 index. Model-implied stock returns are based on eq. (a.12). On Panel (b), the black solid line corresponds to the part of annualized consumption growth that is accounted for by x_t and y_t, that is $6 \times (\mu_{c,0} + \mu_{c,x} x_t + \mu_{c,y} y_t)$ (see eq. 4). Observed annualized consumption is at the quarterly frequency (gray solid line); to obtain a bi-monthly series (dotted gray line), we apply a cubic spline to the quarterly series of consumption.

Figure 5: Model-implied distribution of consumption growth



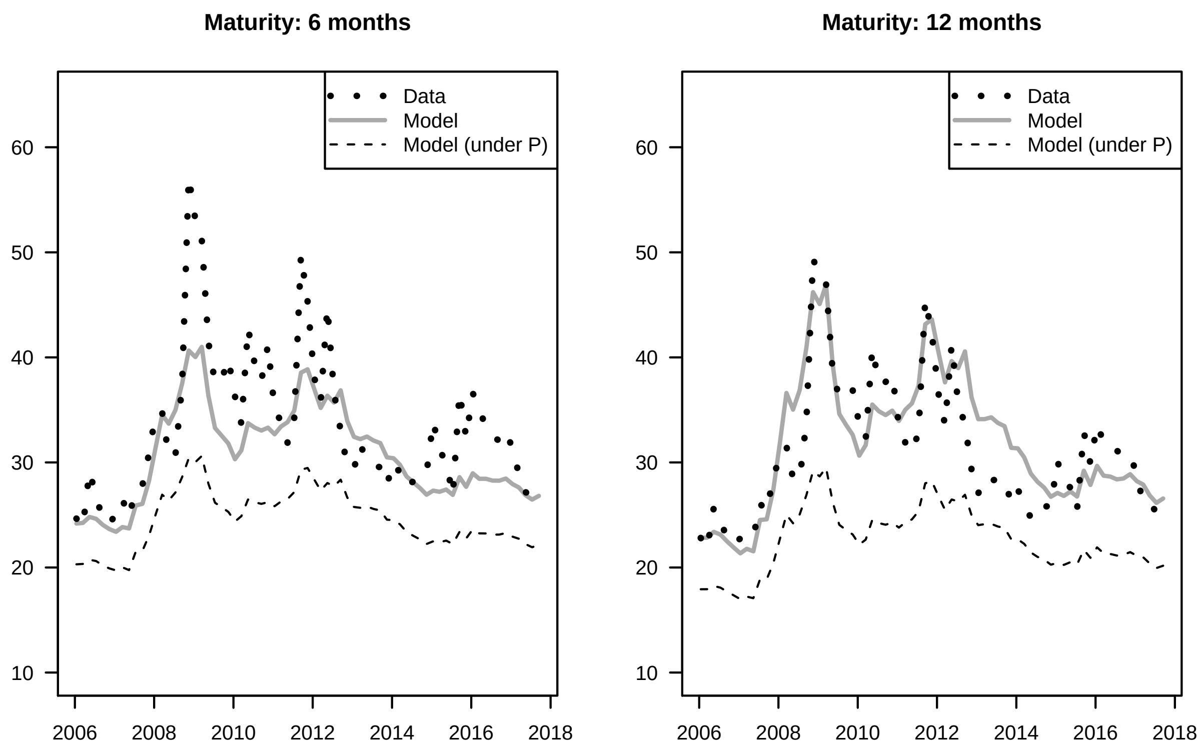
This figure displays the model implied probability density function [p.d.f., Panel (a)] and cumulative distribution function [c.d.f., Panel (b)] of annual (log) consumption growth (black solid lines). On Panel (a), we also report a kernel-based estimate of annual consumption growth based on the Area-wide-Model (AWM) database for the euro area (Fagan et al., 2001, updated database covering the period from 1970Q1 to 2017Q4). The dashed grey line displayed on Panel (b) corresponds to the c.d.f. that would prevail if we had no direct effect of defaults on consumption, that is if we had $\mu_{c,w} = 0$ (se eq. 4). The computation of the c.d.f. of consumption growth is based on a straightforward adaptation of Corollary 2 (using the fact that annual consumption growth is an affine combination of six consecutive values of the state vectors X_t, see eq. 4).

Figure 6: Fit of iTraxx index swap spreads



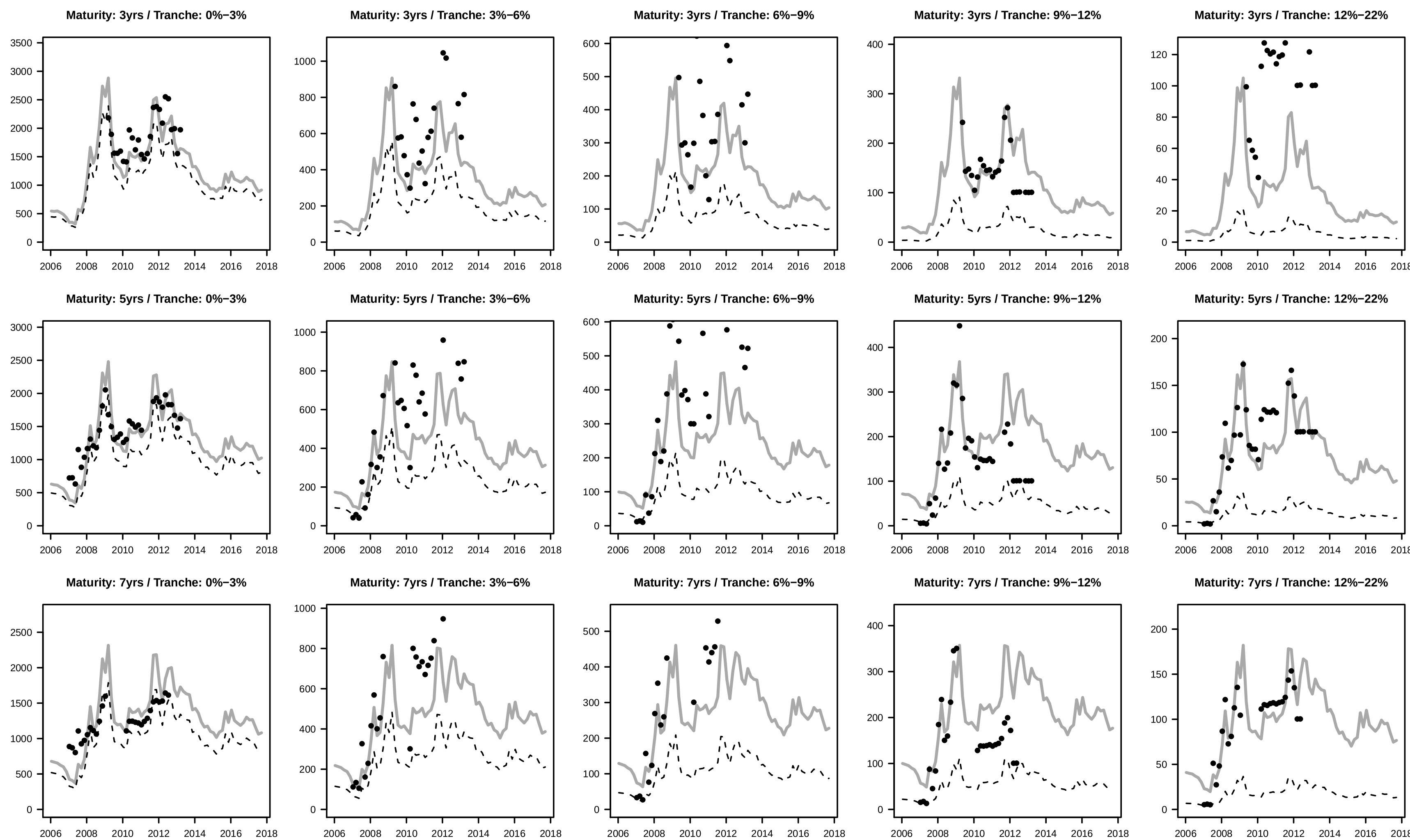
This figure displays index swap spreads (iTraxx Europe main index, black dots) and their model-implied counterparts (gray solid lines). The data cover the period from January 2006 to September 2017 at a bi-monthly frequency. Spreads are expressed in basis points.

Figure 7: Equity options



This figure displays implied volatilities of put options written on the EURO STOXX 50 index (black dots) and their model-implied counterparts (gray solid lines). The dashed black lines represent (model-based) implied volatilities that would prevail if agents were risk-neutral; they correspond to the implied volatilities computed under the physical measure $\mathbb{P}$.

Figure 8: Fit of tranche prices



This figure displays model-implied iTraxx tranche prices (in gray). Black dots represent observed market prices. The dashed black lines correspond to counter-factual tranche prices that would be observed if agents were risk-neutral (i.e. "under $\mathbb{P}$"). The differences between the gray and black lines are credit risk premiums. All prices are expressed in basis points. Lines correspond to a given maturity and columns correspond to a given seniority level. For each date, maturity and tranche, we convert all quotes into an equivalent running spread with no upfront payment by using the risky duration approach (see e.g. O'Kane and Sen, 2003; D'Amato and Gyntelberg, 2005; Morgan Stanley, 2011).

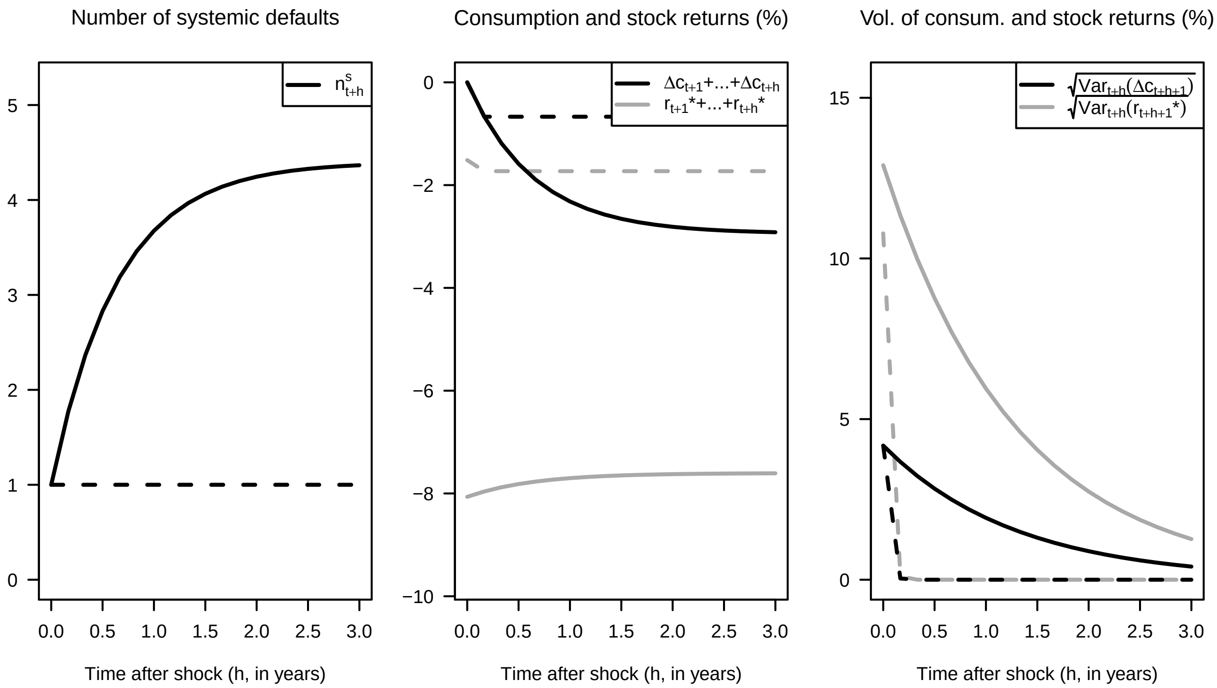
5.2 Dynamic effects of systemic defaults

This subsection examines the dynamic implications of a systemic default. We focus on both consumption and stock returns; the implications on credit derivative prices will be considered in the next subsection. The dynamic analysis relies on impulse response functions (IRFs), where the initial shock consists of an unexpected additional default by a systemic entity. Figure 9 displays the results.

The left-hand side panel of Figure 9 shows the dynamic responses of the number of systemic defaults following an unexpected systemic default on date $t = 0$. Because of contagion phenomena, the initial default increases the expected number of subsequent systemic defaults. More precisely, a systemic default triggers two additional systemic defaults in the subsequent two years, on average. The middle panel shows, in black, the response of consumption following the systemic default. This response is gradual, going from 0 to -3% in the two years following the shock. The economic impact of a systemic default is therefore substantial. For the sake of comparison, using international data, Laeven and Valencia (2012) find that a systemic banking crisis is, on average, followed by a 23% decrease in output, which would correspond to about 8 defaults of systemic entities (assuming that consumption and GDP move in tandem). Interestingly, in our model, a systemic default has not only an impact on conditional expectations, but also on conditional variances: upon arrival of a systemic default, we observe a jump of the volatility of consumption growth, i.e. a dramatic increase in economic uncertainty (right-hand plot of Figure 9).

The middle and right-hand side panels of Figure 9 further display the respective responses of ex-dividend stock returns r_t^* and their volatility. Following a systemic default, the conditional level and volatility of the stock index undergo the same types of effects as consumption does, except that the stock price response (in level, central plot) is immediate. This is consistent with the forward-looking nature of stock returns. The figure also displays (in dotted lines) the responses to an unexpected default of a systemic entity when the contagion channel is switched off. As expected, no contagion implies that the default of a systemic firm will not induce further systemic defaults. Moreover, under no contagion, the effects on consumption and stock returns become muted; dropping by less than 1% and 2%, respectively.

Figure 9: Responses to an unexpected default of a systemic entity



This figure displays response functions of different variables to an additional default of a systemic entity at date $t = 0$. That is, the initial shock is $n^s_{t=0} = \mathbb{E}(n^s_t) + 1$. The left-hand side panel displays the effect on the number of systemic defaults. The middle panel displays changes in expectations of future consumption and of the future stock index. The right-hand side panel shows the effect on the expectations of future conditional variances of consumption growth and of stock returns. To facilitate the reading, we plot the square roots of the expected conditional variance. On each panel, the dashed lines indicate the responses that would prevail in the absence of contagion effects (i.e. if $c_1 = c_2 = 0$, see eq. 2).

5.3 Credit risk premiums

Let us now turn to the study of credit risk premiums, defined as the differences between model-implied prices and those prices that would be observed if agents were not risk averse. The latter prices are computed by replacing $\mathbb{E}^{\mathbb{Q}}$ by $\mathbb{E}^{\mathbb{P}} \equiv \mathbb{E}$ in the pricing formulas. In other words, these counterfactual prices are computed "under the physical measure $\mathbb{P}$" (whereas standard model-implied prices are computed under the risk-neutral measure $\mathbb{Q}$).

Let us first consider the decompositions of Credit Default Swap (CDS) spreads. Figure 10 displays the $\mathbb{P}$ (gray) and $\mathbb{Q}$ (black) CDS spreads associated with maturities of 5 and 10 years. The differences between the two types of spreads are credit risk premiums. The solid lines correspond to spreads of CDSs written on systemic entities. In late 2011, CDS premiums accounted for almost 75% of the 10-year CDS spread. Such high risk premiums reflect the fact that the default of a systemic entity is a particularly bad state of the world, i.e. a state of high marginal utility: when it happens, agents dramatically revise their future consumption path downward (consistently with the IRF plotted on the middle panel of Figure 9). For a CDS written on a systemic entity, the protection seller expects to face large losses in bad states of the world. As a result, she is willing to provide this protection only if the compensation is high enough, i.e. if the CDS spread is sufficiently above her expected loss, which translates into high credit risk premiums.

At this stage, we have not discussed the parametrization of the number of non-systemic defaults ($n_{3,t}$). Given this number does not cause any other variable in the model (see Figure 1), it does not affect the prices we have considered until now. In particular, it was not necessary to parametrize the conditional distribution of $n_{3,t}$ to estimate the model. Thus, we are now free to choose the exposure of non-systemic entities, which we will exploit to investigate the influence of default risk exposure to credit risk premiums. Recall that the conditional distribution of the number of defaults in segment j is $\mathscr{P}(\beta_j y_{t+1} + c_j n_t^s)$ (this is eq. 2). Therefore, the default risk exposure of segment-3 entities is defined by the pair (β_3, c_3). The triangles in Figure 10 are obtained by taking $c_3 = 0.5c_1 = 0.5c_2$ and by computing β_3 in order to have segment-3 entities featuring the same average default probability as the systemic entities (from segments 1 and 2). Figure 10 shows that the spreads of CDS written on these entities are far lower than those for systemic entities. This figure also shows that the $\mathbb{P}$ components of the CDS spreads of systemic entities and segment-3 entities are close. This was expected as $\mathbb{P}$-CDS spreads essentially reflect default probabilities and segment-3 entities have, on average, the same default probability as systemic entities. The reason why credit risk premiums are far lower for segment-3 entities is that the defaults of such entities tend to occur in *relatively* better states of the world than is the case for systemic entities. Though defaults of segment-3 entities are more likely to happen when y_t is high, the decline in consumption may then remain subdued as long as such a high level of y_t has not triggered (recessionary) defaults of systemic entities.

Again, the exposure (β_3, c_3) chosen for segment-3 entities was arbitrary. Another exposure (β_3, c_3) would have resulted in different dotted lines in Figure 10. In particular, we could have chosen $\beta_3 < \beta_1$ and $c_3 > c_1$ (say), still keeping the average default probability constant. In this case, compared to systemic entities, a larger fraction of defaults of segment-3 entities would take place in particularly bad states of the world. Accordingly, we would expect higher CDS spreads for this new type of entities than for the systemic ones. They would remain "non systemic" because their default would still not cause a drop in consumption growth or other defaults.

Let us define the $\mathbb{Q}/\mathbb{P}$ ratio as the ratio between model-implied CDS spreads and counter-factual $\mathbb{P}$-CDS spreads. Figure 11 explores in a systematic way the relationship between the exposures to the risk factors (β_3, c_3) on the one hand, and the 10-year-maturity $\mathbb{Q}/\mathbb{P}$ ratio on the other hand. On Figure 11, we connect, with black lines, those pairs of exposures resulting in the same average $\mathbb{Q}/\mathbb{P}$ ratio. We also connect, with dashed gray lines, pairs of exposures resulting in the same average one-year probability of default. Whereas the black square represents the segment-3 entity we considered in Figure 10, the triangle indicates an entity that features the same exposures as our systemic entities. While the average default probabilities of these two types of entities are close, their $\mathbb{Q}/\mathbb{P}$ ratios differ substantially (2.75 and 2, respectively). The figure also shows that, for each average probability of default, there exists a maximum $\mathbb{Q}/\mathbb{P}$ ratio. Typically, for a one-year probability of default of 0.2% (say), the maximum $\mathbb{Q}/\mathbb{P}$ ratio is about 3.

Credit risk premiums are also present in iTraxx tranche spreads. On Figure 8, these risk premiums are the differences between the gray lines and the dashed black lines: while the gray lines are the model-implied tranche prices, the dotted lines are their $\mathbb{P}$-counterparts, i.e. the (model-implied) prices that would prevail if agents were not risk averse. The more senior the tranche, the higher the relative importance of credit risk premiums. This is consistent with the fact that more senior tranches are more exposed to catastrophic events.

5.4 Measuring systemic risk

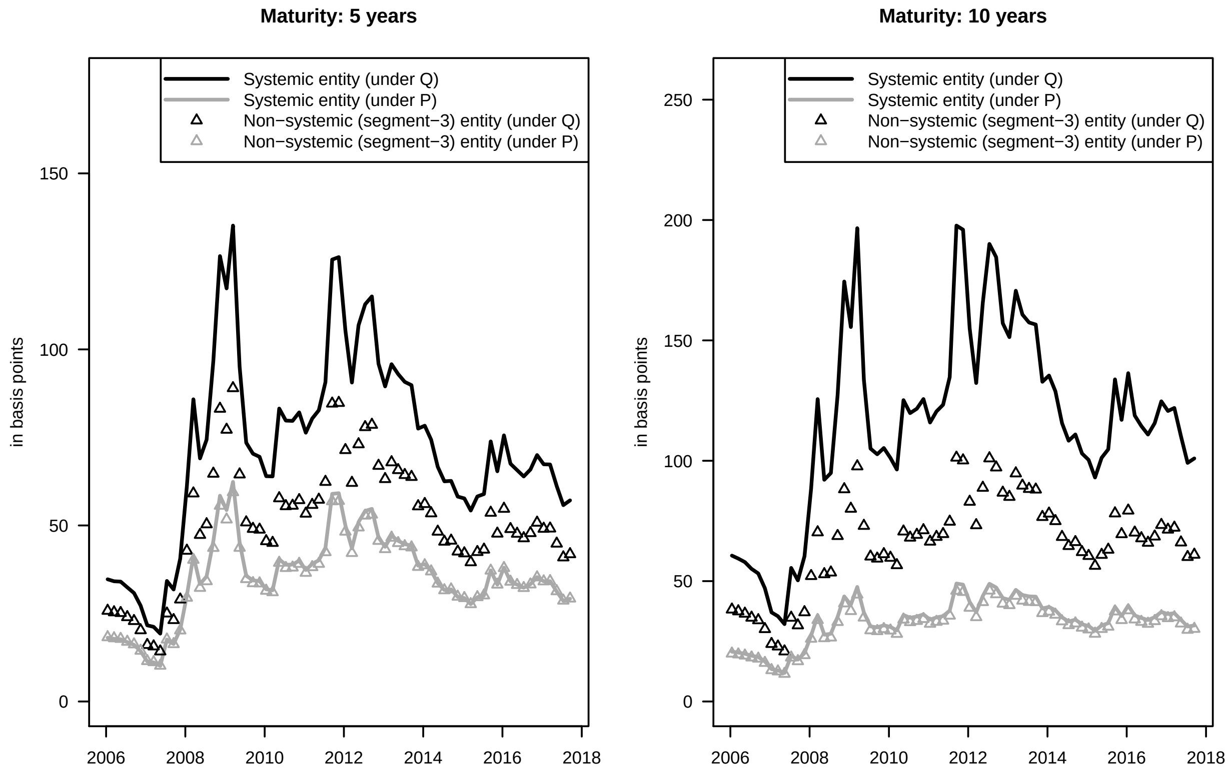
Our approach provides natural measures of systemic risk by considering the probabilities of having at least ω systemic defaults (say) at any given horizon h.[31]

As an illustration, the top panel of Figure 12 plots the probability of observing at least 10 iTraxx constituents defaulting in the next 12 months (dotted line) and 24 months (solid line). We also report vertical lines indicating significant dates of the financial crisis. Our systemic indicators reached their maximum levels in late 2008, after the Lehman bankruptcy and in late 2011, when the European sovereign crisis was at its peak. For these two dates, the conditional probabilities to have more than 10 defaults among iTraxx constituents within two years were of 7% and 6%, respectively.

The bottom panel of Figure 12 depicts an alternative indicator of systemic risk by reporting the probabilities of consumption dropping by more than 10% (solid line) and 20% (dotted line) in the next 12 months. In line with our first indicator, these probabilities reached their maximum levels at the time of the Lehman bankruptcy and at the height of the European sovereign crisis. On these two dates, the probabilities of consumption dropping by more than 10% in the next 12 months were of 6% and 5%, respectively.

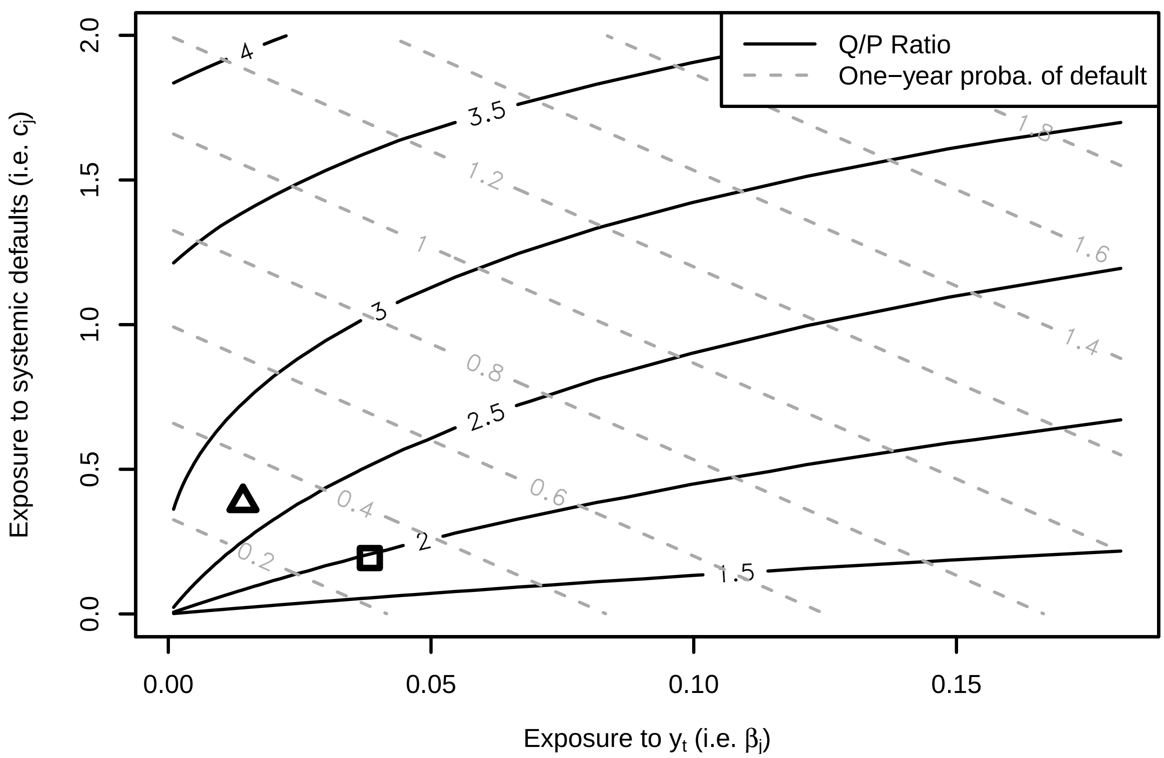
[31] Closed-form formulas can be deduced from a straightforward adaptation of Corollary 2.

Figure 10: Credit risk premiums in iTraxx Europe main indexes



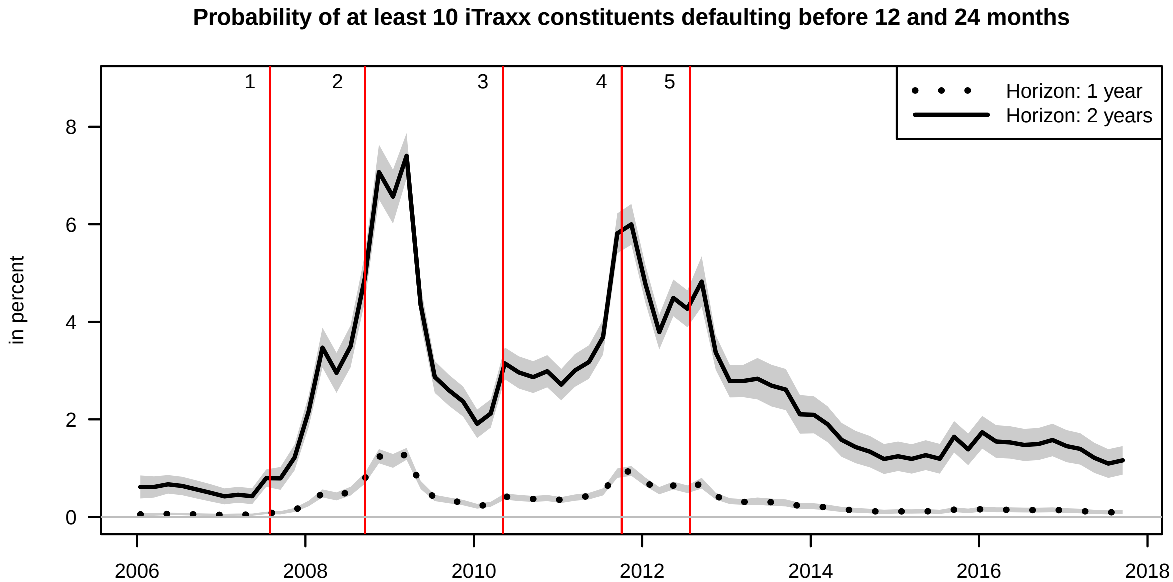
This figure illustrates the magnitude of credit risk premiums in iTraxx Europe main indexes. The black solid line is the model-implied iTraxx index. The gray solid line is the (counter-factual) iTraxx index that would prevail if agents were not risk averse (said to be the iTraxx index "under the physical measure $\mathbb{P}$"). The difference between the black and gray solid lines reflects credit risk premiums. Triangles correspond to ($\mathbb{P}$ and $\mathbb{Q}$) CDS spreads associated with a firm from the third segment; these entities are non-systemic, in the sense that their defaults do not Granger-cause consumption; however, they may be exposed to systemic-risk factors, which implies that the CDS written on segment-3 entities may embed substantial risk premiums. Recall that the conditional distribution of number of defaults in segment j is $\mathscr{P}(\beta_j y_{t+1} + c_j n_t^s)$ (see eq. 2). As explained in Subsection 5.3, the specification of the exposure of segment-3 entities (i.e. β_3 and c_3) is arbitrary. In the context of this chart, the exposure is defined by $c_3 = 0.5 \times c_1 = 0.5 \times c_2$, and by setting β_3 in such a way as to have the same average default probability as systemic entities (segments 1 and 2). This is why $\mathbb{P}$ CDS spreads coincide for systemic and segment-3 entities. This is not the case for $\mathbb{Q}$ CDS spreads: because $c_3 < c_1 = c_2$, segment-3 entities are relatively less exposed to disasters than systemic entities, which translates into lower credit-risk premiums. See Subsection 5.3 for more details.

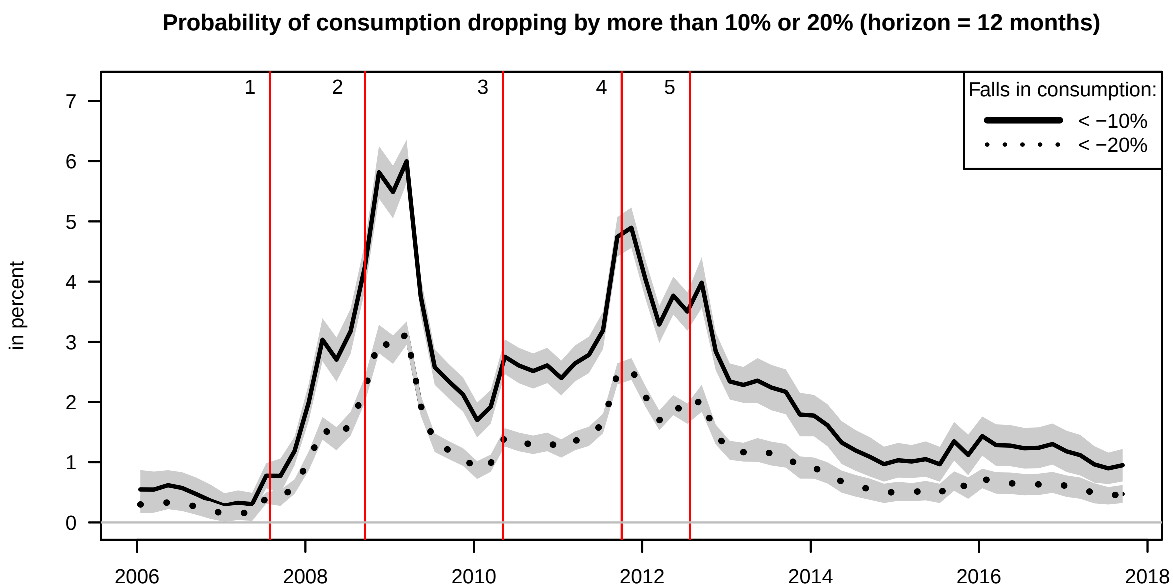
Figure 11: Impact of exposures to the exogenous factor y_t (measured by β_j) and to the number of systemic defaults n_t^s (measured by c_j) on the average size of credit risk premiums



This figure illustrates the influence of the exposure to the risk factors – that are the exogenous variable y_t and the number of systemic defaults n_t^s – on the relative importance of risk premiums in CDS spreads. The coordinates of each point correspond to the exposure of a given entity to factor y_t (abscissa) and to the number of systemic defaults, i.e. n_t^s (ordinate). The black lines connect the pairs of exposures implying the same $\mathbb{Q}/\mathbb{P}$ ratio, defined as the ratio between the (model-implied) CDS spread and the counter-factual CDS spread that would be observed if agents were risk-neutral. (The former is the one computed under the pricing, or risk-neutral, measure $\mathbb{Q}$; the latter is computed under the physical measure $\mathbb{P}$, hence the denomination "$\mathbb{Q}/\mathbb{P}$ ratio".) We consider the 10-year maturity. The gray dashed lines connect pairs of exposures implying the same average probability of default. Figures reported in gray are probabilities of default expressed in annualized percentage points. The triangle indicates a pair of exposures corresponding to the systemic entities. The square indicates the pair of exposures of non-systemic entities whose CDS indexes are displayed in Figure 10.

Figure 12: Systemic indicators





The top panel of this figure displays the (model-implied) probabilities that at least 10 iTraxx constituents – considered to be systemic entities – default in the coming 12 months (dotted line) and 24 months (solid line). The bottom panel displays the (model-implied) probabilities that consumption drops by at least 10% (solid line) and 20% (dotted line) in the next 12 months. Gray-shaded areas are 95% confidence bands; they reflect the uncertainty surrounding filtered x_t and y_t. The vertical bars correspond to important dates of the financial crisis (see Bruegel, http://bruegel.org/2015/09/euro-crisis/): (1) August 2007: European interbank markets seize-up; (2) 15 September 2008: Collapse of Lehman Brothers; (3) 7 May 2010: Emergency measures to safeguard financial stability; (4) October 2011: Spain and Italy are hit by a wave of rating downgrades by the three main rating agencies; (5) 26 July 2012: ECB President Mario Draghi says that the ECB will do "whatever it takes to preserve the euro".

6 Concluding remarks

In the literature on disaster risk and its asset pricing implications, disasters are usually modeled as exogenous shocks that simultaneously impact macroeconomic variables and default probabilities of firms. In the present paper, we propose a no-arbitrage asset pricing framework that allows for the default of large (systemic) firms to constitute a disaster in itself. More precisely, in our model, such a systemic default (i) has an adverse macroeconomic impact, and (ii) can trigger default cascades; thus leading to an amplification of the original recessionary effect.

Before the occurrence of such a disaster, the probabilities of default depend on a small number of factors capturing the fluctuations of the economy. Risk-averse investors observe these factors, and price derivatives that are consistent with the model structure. The model offers closed-form solutions for a wide variety of derivatives that are exposed to systemic risk, notably far-OTM equity puts and credit derivatives. In this context, the econometrician can infer the model parametrization from observed prices even if no systemic default has taken place over the estimation period. That is because, typically, prices of CDOs provide information on the views of investors regarding contagion effects. Indeed, the parameters governing the contagion mechanism influence the conditional distribution of the number of future defaults, which CDO prices directly depend on.

The empirical application, on euro-area data, demonstrates the ability of our model to capture a substantial share of the joint fluctuations of stock and credit markets, both in tranquil and stressed periods. Our approach also provides an estimation of the conditional effect of a systemic default on consumption. This effect is found to be substantial: the default of a systemic entity is expected to be followed by a 3% decrease in consumption. We finally derive systemic risk indicators defined as: (i) the probability of joint defaults of systemic entities, and (ii) the probability of consumption displaying a sharp drop in the next year. In addition, our results point to the existence of important risk premiums in the credit derivatives written on systemic entities. This implies that simple inversions of CDS pricing formulas to obtain probabilities of default – without correcting for risk aversion – may tend to substantially overestimate these probabilities.

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A State-vector dynamics

A.1 The dynamics of $z_t = [x_t, y_t]'$

We assume that, conditional on $\{\underline{x_t}, \underline{y_t}\}$, x_{t+1} and y_{t+1} are independently drawn from non-centered Gamma distributions:[32]

$$\begin{aligned} x_{t+1}|\underline{x_t}, \underline{y_t} &\sim \gamma_{\nu_x}(\zeta_x x_t, \mu_x) \\ y_{t+1}|\underline{x_t}, \underline{y_t} &\sim \gamma_{\nu_y}(\zeta_{y,x} x_t + \zeta_{y,y} y_t, \mu_y). \end{aligned}$$

In this case, we have that (Monfort et al., 2017):

$$\begin{aligned} x_t &= \mu_x \nu_x + \mu_x \zeta_x x_{t-1} + \sigma_{x,t} \varepsilon_{x,t} \\ y_t &= \mu_y \nu_y + \mu_y \zeta_{y,x} x_{t-1} + \mu_y \zeta_{y,y} y_{t-1} + \tilde{\sigma}_{y,t} \tilde{\varepsilon}_{y,t}, \end{aligned}$$

where $\tilde{\varepsilon}_t = [\varepsilon_{x,t}, \tilde{\varepsilon}_{y,t}]'$ is a martingale difference sequence with identity covariance matrix and where:

$$\begin{aligned} \sigma_{x,t} &= \mu_x \sqrt{\nu_x + 2\zeta_x x_{t-1}}, \\ \tilde{\sigma}_{y,t} &= \mu_y \sqrt{\nu_y + 2\zeta_{y,y} y_{t-1} + 2\zeta_{y,x} x_{t-1}}. \end{aligned}$$

Let us use the notations $\rho_x = \mu_x \zeta_x$ and $\rho_y = \mu_y \zeta_{y,y}$ and let us assume that (i) $1 - \rho_x = \mu_x \nu_x = \mu_y \nu_y$ and that (ii) $\rho_x - \rho_y = \mu_y \zeta_{y,x}$. We get:

$$\left\{ \begin{aligned} x_t - 1 &= \rho_x (x_{t-1} - 1) + \sigma_{x,t} \varepsilon_{x,t}, \\ y_t &= 1 - \rho_x + \rho_x x_{t-1} + \rho_y (y_{t-1} - x_{t-1}) + \tilde{\sigma}_{y,t} \tilde{\varepsilon}_{y,t}. \end{aligned} \right.$$

Defining $\varepsilon_{y,t} = \dfrac{\tilde{\sigma}_{y,t} \tilde{\varepsilon}_{y,t} - \sigma_{x,t} \varepsilon_{x,t}}{\sqrt{\tilde{\sigma}_{y,t}^2 + \sigma_{x,t}^2}}$ and $\sigma_{y,t} = \sqrt{\tilde{\sigma}_{y,t}^2 + \sigma_{x,t}^2}$ leads to System (1).

[32] The random variable W is drawn from a non-centered Gamma distribution $\gamma_\nu(\varphi, \mu)$, iif there exists a $\mathscr{P}(\varphi)$-distributed variable Z such that $W|Z \sim \gamma(\nu + Z, \mu)$ where Z and μ are, respectively, the shape and scale parameters of the Gamma distribution (Gouriéroux and Jasiak, 2006, see e.g.). When $Z = 0$ and $\nu = 0$, then $W = 0$. When $\nu = 0$, this distribution is called Gamma$_0$ distribution; this case is introduced and studied by Monfort et al. (2017).

A.2 The conditional log-Laplace transform of $X_t = [x_t, y_t, w_t, N_t', N_{t-1}']'$

The dynamics followed by $X_t = [x_t, y_t, w_t, N_t', N_{t-1}']'$ is a special case of the general case treated in the Online Appendix (see O.1) with:

$$\zeta_F = \begin{bmatrix} \zeta_x & \zeta_{y,x} & 0 \\ 0 & \zeta_{y,y} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \zeta_n = \begin{bmatrix} 0 & 0 & \xi_w \\ 0 & 0 & \xi_w \\ 0 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\mu = [\mu_x, \mu_y, \mu_w]'$, $\nu = [\nu_x, \nu_y, 0]'$.

As shown in the Online Appendix (O.1), in this case the conditional log Laplace transform of X_t is given by:

$$\mathbb{E}_t\left(\exp(v'X_{t+1})\right) = \exp(\psi(v, X_t)) = \exp(\psi_0(\nu) + \psi_1(\nu)'X_t), \tag{a.1}$$

where

$$\begin{cases} \psi_0(v) &= d\left(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A\right), \\ \psi_1(v) &= [\psi_{1,A}(v)', \psi_{1,B}(v)', \psi_{1,C}(v)']', \end{cases} \tag{a.2}$$

with $d(w) = -\nu' \log(1 - w \odot \mu)$, where $\odot$ is the element-by-element (Hadamard) product (and where, by abuse of notations, the log operator is applied element-by-element wise) and where:

$$\begin{cases} \psi_{1,A}(v) &= a(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta^j + v_A), \\ \psi_{1,B}(v) &= b(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta^j + v_A) + \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)c^j + v_B + v_C, \\ \psi_{1,C}(v) &= c(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta^j + v_A) - \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)c^j, \end{cases} \tag{a.3}$$

where β^j and c^j denote the j^{th} columns of β and of c respectively, and where $v = [v_A', v_B', v_C']'$, where v_A is a n_F-dimensional vector and v_B and v_C are J-dimensional vectors, and:

$$a(w) \;=\; \zeta_F\left(\frac{w \odot \mu}{1 - w \odot \mu}\right), \quad b(w) = \zeta_n\left(\frac{w \odot \mu}{1 - w \odot \mu}\right), \quad c(w) = -b(w),$$

where, again, the log and division operator are applied element wise, by abuse of notations.

B Derivation of the s.d.f.

Proposition 1. *$u_t = u_{t-1} + \mu_{c,0} + \mu'_{u,1}X_t + (\mu_{c,1} - \mu_{u,1})'X_{t-1} + (1-\delta)\sigma_c\varepsilon_t^c$ satisfies eq. (6) for any $[X'_t, X'_{t-1}]'$ iff $\mu_{u,1}$ satisfies:*

$$\frac{\delta}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) = \mu_{u,1} - \mu_{c,1}, \tag{a.4}$$

where $\mu_{c,1} = [\mu_{c,x}, \mu_{c,y}, \mu_{c,w}, \mathbf{0}']'$ (see eq. 4).

Proof. Let us consider the following specification for Δu_t:

$$\Delta u_t = \mu_{u,0} + \mu'_{u,1}X_t + \mu'_{u,2}X_{t-1} + (1-\delta)\sigma_c\varepsilon_t^c.$$

Then, for a given $[X'_t, X'_{t-1}]'$, we have:

$$\begin{aligned}\text{eq. (6)} \quad \Leftrightarrow \quad & \mu_{u,0} + \mu'_{u,1}X_t + \mu'_{u,2}X_{t-1} \\ & = \mu_{c,0} + \mu'_{c,1}X_t + \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\left\{[\psi_1((1-\gamma)\mu_{u,1}) + (1-\gamma)\mu_{u,2}]'(X_t - X_{t-1})\right\}.\end{aligned}$$

Therefore eq. (6) is satisfied for any $[X'_t, X'_{t-1}]'$ iff

$$\left\{\begin{array}{l}\frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) + \frac{1}{1-\delta}\mu_{u,2} = 0,\\ \mu_{u,1} - \mu_{c,1} - \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) - \frac{\delta}{1-\delta}\mu_{u,2} = 0,\\ \mu_{u,0} = \mu_{c,0},\end{array}\right.$$

or

$$\left\{\begin{array}{l}\frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) + \frac{1}{1-\delta}\mu_{u,2} = 0,\\ \mu_{u,1} + \mu_{u,2} - \mu_{c,1} = 0,\\ \mu_{u,0} = \mu_{c,0},\end{array}\right. \tag{a.5}$$

which leads to the result. □

Proposition 2. *We have:*

$$M_{t,t+1} = \exp\left[-(\eta_0 + \eta'_1X_t) + \pi'X_{t+1} - \psi(\pi, X_t) - \eta_c\varepsilon_t^c - \frac{1}{2}\eta_c^2\right], \tag{a.6}$$

with

$$\begin{cases} \pi &= (1-\gamma)\mu_{u,1} - \mu_{c,1} \\ \eta_0 &= -\log(\delta) + \mu_{c,0} + \psi_0((1-\gamma)\mu_{u,1}) - \psi_0(\pi) + (1-\gamma)(1-\delta)\sigma_c^2 \\ \eta_1 &= \psi_1((1-\gamma)\mu_{u,1}) - \psi_1(\pi) \\ \eta_c &= (\gamma(1-\delta)+\delta)\sigma_c \end{cases}$$

Proof. When agents' preferences are as in eq. (6), the s.d.f. is given by (Piazzesi and Schneider, 2007):

$$M_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \frac{\exp[(1-\gamma)u_{t+1}]}{\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}])}.$$

Therefore, we have:

$$\begin{aligned} \log M_{t,t+1} &= \log\delta - \Delta c_{t+1} + (1-\gamma)u_{t+1} - \log\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}]) \\ &= \log\delta - \Delta c_{t+1} + (1-\gamma)\Delta u_{t+1} - \log\mathbb{E}_t(\exp[(1-\gamma)\Delta u_{t+1}]) \\ &= \log(\delta) - \mu_{c,0} - \mu_{c,1}'X_{t+1} - \sigma_c\varepsilon_t^c + (1-\gamma)(\mu_{u,0} + \mu_{u,1}'X_{t+1} + \mu_{u,2}'X_t + (1-\delta)\sigma_c\varepsilon_t^c) \\ &\quad - \log\mathbb{E}_t(\exp[(1-\gamma)(\mu_{u,0} + \mu_{u,1}'X_{t+1} + \mu_{u,2}'X_t + (1-\delta)\sigma_c\varepsilon_t^c)]) \\ &= \log(\delta) - \mu_{c,0} - \psi_0((1-\gamma)\mu_{u,1}) + [(1-\gamma)\mu_{u,1} - \mu_{c,1}]'X_{t+1} - \psi_1((1-\gamma)\mu_{u,1})'X_t \\ &\quad - \sigma_c\varepsilon_t^c + (1-\gamma)(1-\delta)\sigma_c\varepsilon_t^c - \frac{1}{2}(1-\gamma)^2(1-\delta)^2\sigma_c^2, \end{aligned}$$

which leads to the result. □

Proposition 3. *We have:*

$$\exp\left(\psi^{\mathbb{Q}}(v,X_t)\right) \equiv \mathbb{E}_t^{\mathbb{Q}}\left(\exp(v'X_{t+1})\right) = \exp\left(\psi_0^{\mathbb{Q}}(v) + \psi_1^{\mathbb{Q}}(v)'X_t\right), \tag{a.7}$$

with

$$\begin{cases} \psi_0^{\mathbb{Q}}(v) &= \psi_0(v+\pi) - \psi_0(\pi), \\ \psi_1^{\mathbb{Q}}(v) &= \psi_1(v+\pi) - \psi_1(\pi). \end{cases}$$

Proof. Using the s.d.f. specification of Eq. (a.6), we get:

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left(\exp(v'X_{t+1})\right) &= \mathbb{E}_t\left(\frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})}\exp(v'X_{t+1})\right) \\ &= \mathbb{E}_t\left(\exp[\pi'X_{t+1} - \psi(\pi,X_t) + v'X_{t+1}]\right) \\ &= \exp(\psi(v+\pi,X_t) - \psi(\pi,X_t)), \end{aligned}$$

which leads to Eq. (a.7). □

C Pricing formulas

C.1 Generic pricing formulas

C.1.1 Pricing of $\exp(u'X_{t+h})$ and $v'X_{t+h}$, settled at date $t+h$

Proposition 4. *The date-t price $p(u,h,X_t)$ of the payoff $\exp(u'X_{t+h})$, that is settled at date $t+h$, is given by $\exp(\Gamma_{0,h}(u)+\Gamma'_{1,h}(u)X_t)$, where:*

$$\begin{cases} \Gamma_{1,h+1}(u) &= \psi_1^{\mathbb{Q}}(\Gamma_{1,h}(u)) - \eta_1, \\ \Gamma_{0,h+1}(u) &= \psi_0^{\mathbb{Q}}(\Gamma_{1,h}(u)) - \eta_0 + \Gamma_{0,h}(u), \end{cases}$$

with $\Gamma_{1,0}(u)=u$ and $\Gamma_{0,0}(u)=0$.

Proof. This proposition is clearly satisfied for $h=0$. Assume that, for a given $h\geq 0$ and for all admissible u and X_t, we have $p(u,h,X_t)=\exp(\Gamma_{0,h}(u)+\Gamma_{1,h}(u)'X_t)$, then

$$\begin{aligned} p(u,h+1,X_t) &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(p(u,h,X_{t+1})) \\ &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(\Gamma_{0,h}(u)+\Gamma_{1,h}(u)'X_{t+1})) \\ &= \exp(-\eta_0+\Gamma_{0,h}(u)-\eta_1'X_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(\Gamma_{1,h}(u)'X_{t+1})) \\ &= \exp(-\eta_0+\Gamma_{0,h}(u)-\eta_1'X_t)\exp\left(\psi_0^{\mathbb{Q}}(\Gamma_{1,h}(u))+\psi_1^{\mathbb{Q}}(\Gamma_{1,h}(u))'X_t\right), \end{aligned}$$

which leads to the result by recursion. □

Corollary 1. *The date-t price of the payoff $v'X_{t+h}$, conditional on $X_t=x$, with payoff settlement at date $t+h$, is given by:*

$$\Pi(v,h,x) = v'\nabla_u p(u,h,x)\big|_{u=0}, \tag{a.8}$$

where $p(u,h,x)$ is defined in Proposition 4 and where ∇_u denotes the Jacobian operator with respect to the first argument of the function.

Let us denote by $\mathbf{0}_{r\times c}$ and $\mathbf{1}_{r\times c}$ the matrices of dimensions $r\times c$ filled with 0 and 1, respectively. In addition, let e_j denote the j^{th} row vector of the identity matrix of dimension $J\times J$. Using the previous corollary with $v'=[\mathbf{0}_{1\times 3},e_j,\mathbf{0}_{1\times J}]$ and $v'=[\mathbf{0}_{1\times 3},\mathbf{0}_{1\times J},e_j]$, respectively, results in the prices of the payoffs $N_{j,t+h}$ and $N_{j,t+h-1}$, settled at date $t+h$.

Corollary 2. *The date-t price of the payoff $\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}$, conditional on $X_t=x$, with*

payoff settlement at date $t+h$, is given by:

$$\begin{aligned} g(a,b,y,h,x) &= \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}|X_t=x\right) \\ &= \frac{p(a,h,x)}{2} - \frac{1}{\pi}\int_0^{\infty}\frac{Im\left[p(a+ivb,h,x)\exp(-ivy)\right]}{v}dv, \end{aligned} \tag{a.9}$$

where $Im(z)$ denotes the imaginary part of the complex number z and where $\lambda_{t,t+h}$ is defined in eq. (10).

This result is proved in Duffie et al. (2000). Note that the formula for $g(a,b,y,h,x)$ is quasi explicit since it only involves a simple (one-dimensional) integration.

Corollary 3. *The date-t price of the payoff $a'X_{t+h}\mathbb{1}_{\{b'X_{t+h}<y\}}$, conditional on $X_t=x$, with payoff settlement at date $t+h$, is given by:*

$$\Gamma(a,b,y,h,x) = a'\nabla_u g(u,b,y,h,x)\big|_{u=0}. \tag{a.10}$$

Let us consider the date-t prices of the following payoffs, settled at date $t+k$: (i) $\mathbb{1}_{\{N_{1,t+k}<z\}}$, (ii) $N_{1,t+k}\mathbb{1}_{\{N_{1,t+k}<z\}}$ and (iii) $N_{1,t+k-1}\mathbb{1}_{\{N_{1,t+k}<z\}}$. Using the notations introduced in Corollaries 2 and 3, these prices are respectively equal to: (i) $g(0,\omega_0,z,h,X_t)$, (ii) $\Gamma(\omega_0,\omega_0,z,h,X_t)$ and (iii) $\Gamma(\omega_1,\omega_0,z,h,X_t)$, with $\omega_0=[\mathbf{0}_{1\times3},e_1,\mathbf{0}_{1\times J}]'$ and $\omega_1=[\mathbf{0}_{1\times3},\mathbf{0}_{1\times J},e_1]'$, where e_1 is a J-dimensional vector whose entries are 0, except the first one that is equal to 1.

C.1.2 Pricing of $\exp(u_1'X_{t+1}+\cdots+u_1'X_{t+h-1}+u_2'X_{t+h})$, settled at date $t+h$

Proposition 5. *Using the notation $\mathbf{u}=\{u_1,u_2\}$, the date-t price $\tilde{p}(\mathbf{u},h,X_t)$ of the payoff*

$$\exp(u_1'X_{t+1}+\cdots+u_1'X_{t+h-1}+u_2'X_{t+h}), \quad \text{for } h>1$$

and of $\exp(u_2'X_{t+1})$ for $h=1$, settled at date $t+h$, is given by $\exp\left(\tilde{\Gamma}_{0,h}(\mathbf{u})+\tilde{\Gamma}_{1,h}(\mathbf{u})'X_t\right)$, where:

$$\begin{cases} \tilde{\Gamma}_{1,h+1}(\mathbf{u}) &= \psi_1^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(\mathbf{u})+u_1)-\eta_1, \\ \tilde{\Gamma}_{0,h+1}(\mathbf{u}) &= \psi_0^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(\mathbf{u})+u_1)-\eta_0+\tilde{\Gamma}_{0,h}(\mathbf{u}), \end{cases}$$

with $\tilde{\Gamma}_{1,1}(\mathbf{u})=\Gamma_{1,1}(u_2)$ and $\tilde{\Gamma}_{0,1}(\mathbf{u})=\Gamma_{0,1}(u_2)$.

Proof. This proposition is clearly satisfied for $h=1$. Assume that, for a given $h\geq1$ and for all admissible $\mathbf{u}$ and X_t, we know $\exp\left(\tilde{\Gamma}_{0,h}(\mathbf{u})+\tilde{\Gamma}_{1,h}(\mathbf{u})'X_t\right)$, then

$$\begin{aligned}\tilde{p}(\mathbf{u},h+1,X_t) &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(u_1'X_{t+1})\tilde{p}(\mathbf{u},h,X_{t+1}))\\ &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(\tilde{\Gamma}_{0,h}(\mathbf{u})+\tilde{\Gamma}_{1,h}(\mathbf{u})'X_{t+1}+u_1'X_{t+1}))\\ &= \exp(-\eta_0+\tilde{\Gamma}_{0,h}(\mathbf{u})-\eta_1'X_t)\mathbb{E}_t^{\mathbb{Q}}(\exp([\tilde{\Gamma}_{1,h}(\mathbf{u})+u_1]'X_{t+1}))\\ &= \exp(-\eta_0+\tilde{\Gamma}_{0,h}(\mathbf{u})-\eta_1'X_t)\exp\left(\psi_0^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(\mathbf{u})+u_1)+\psi_1^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(\mathbf{u})+u_1)'X_t\right),\end{aligned}$$

which leads to the result by recursion. □

C.2 Tranche products formula

Let's rewrite eq. (11):

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}\frac{\tilde{N}_{t+k}-\tilde{N}_{t+k-1}}{\bar{b}-\bar{a}}\mathbb{1}_{\{\bar{a}<\tilde{N}_{t+k}\leq\bar{b}\}}\right\}\\ =\;& U_{t,h}^{TDS}(a,b)+\mathbb{E}_t^{\mathbb{Q}}\left\{\frac{S_{t,h}^{TDS}(a,b)}{q}\sum_{k=1}^{qh}\Lambda_{t,t+k}\left(\mathbb{1}_{\{\tilde{N}_{t+k}\leq\bar{a}\}}+\frac{\bar{b}-\tilde{N}_{t+k}}{\bar{b}-\bar{a}}\mathbb{1}_{\{\bar{a}<\tilde{N}_{t+k}\leq\bar{b}\}}\right)\right\},\end{aligned}$$

where $\bar{a}=a\frac{\tilde{I}}{1-RR}$ and $\bar{b}=b\frac{\tilde{I}}{1-RR}$. We obtain:

$$S_{t,h}^{TDS}(a,b) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}(\tilde{N}_{t+k}-\tilde{N}_{t+k-1})\left(\mathbb{1}_{\{\tilde{N}_{t+k}\leq\bar{b}\}}-\mathbb{1}_{\{\tilde{N}_{t+k}\leq\bar{a}\}}\right)\right\}-(\bar{b}-\bar{a})U_{t,h}^{TDS}(a,b)}{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}\left((\bar{b}-\bar{a})\mathbb{1}_{\{\tilde{N}_{t+k}\leq\bar{a}\}}+(\bar{b}-\tilde{N}_{t+k})\left(\mathbb{1}_{\{\tilde{N}_{t+k}\leq\bar{b}\}}-\mathbb{1}_{\{\tilde{N}_{t+k}\leq\bar{a}\}}\right)\right)\right\}}.$$

C.3 Approximated stock returns

Proposition 6. *If the log growth rate of dividends is affine in X_t, i.e. if:*

$$g_{d,t}=\mu_{d,0}+\mu_{d,1}'X_t, \tag{a.11}$$

then stock returns are approximately given by:

$$r_{t+1}^s=\kappa_0+A_0(\kappa_1-1)+\mu_{d,0}+(\kappa_1A_1+\mu_{d,1})'X_{t+1}-A_1'X_t, \tag{a.12}$$

where κ_0 and κ_1 are given by

$$\begin{cases}\kappa_1 = \dfrac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})},\\ \kappa_0 = \log(1+\exp(\bar{\tau}))-\kappa_1\bar{\tau},\end{cases} \tag{a.13}$$

where A_1 *satisfies*

$$\psi_1(\kappa_1 A_1 + \mu_{d,1} + \pi) = A_1 + \eta_1 + \psi_1(\pi),$$

and where

$$A_0 = \frac{-\kappa_0 - \mu_{d,0} + \eta_0 + \psi_0(\pi) - \psi_0(\kappa_1 A_1 + \mu_{d,1} + \pi)}{\kappa_1 - 1}.$$

Proof. Let us introduce the log price-dividend ratio defined by $\tau_t = \log(P_t/D_t)$ and let us denote by $\bar{\tau}$ its marginal expectation. The following lemma is based on the log-linearization proposed by Campbell and Shiller (1988).

Lemma 1. *if* $\tau_t - \bar{\tau}$ *is relatively small, then the stock return* r^s_{t+1} *can be approximated by:*

$$r^s_{t+1} = \log\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right) \approx \kappa_0 + \kappa_1 \tau_{t+1} - \tau_t + g_{d,t+1}. \tag{a.14}$$

Proof. See Online Appendix O.3 □

Assume that τ_t is affine in X_t, i.e.:

$$\tau_t = A_0 + A_1' X_t. \tag{a.15}$$

Substituting for τ_t in eq. (a.14) leads to eq. (a.12). Let us now determine the constraints that should be satisfied by A_0 and A_1. The returns of stocks have to satisfy the Euler equation:

$$0 = \log \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+1} \exp(r^s_{t+1})). \tag{a.16}$$

Using eqs. (7) and (a.12), we obtain:

$$M_{t,t+1} \exp(r^s_{t+1}) = \tag{a.17}$$
$$\exp(\kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} - \eta_0 - \psi_0(\pi) + (\kappa_1 A_1 + \mu_{d,1} + \pi)' X_{t+1} - (A_1 + \eta_1 + \psi_1(\pi))' X_t).$$

eqs. (a.17) and (a.16) are satisfied if:

$$\begin{cases} \kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} - \eta_0 - \psi_0(\pi) + \psi_0(\kappa_1 A_1 + \mu_{d,1} + \pi) & = & 0, \\ \psi_1(\kappa_1 A_1 + \mu_{d,1} + \pi) - (A_1 + \eta_1 + \psi_1(\pi)) & = & 0, \end{cases}$$

which proves Prop. 6. □

C.4 Equity option pricing

If τ_t and $g_{d,t}$ are affine in X_t (as in eqs. (a.11) and (a.15)), then eq. (15) implies that P_{t+h}/P_t is exponential affine in X_t.

Let us introduce function φ defined by:

$$(u,h,X_t) \to \varphi(u,h,X_t) = \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp\left(u\log\left(\frac{P_{t+h}}{P_t}\right)\right)\right).$$

Using eq. (15) and Prop. 5 (see Subsection C.1.2), one can find functions $a_h^s(\bullet)$ and $b_h^s(\bullet)$ such that:

$$\varphi(u,h,X_t) = \exp(a_h^s(u) + b_h^s(u)'X_t).$$

Replacing $p(a,h,x)$ by $\varphi(a,h,x)$ in Corollary 2 of Subsection C.1.1 provides formulas to compute:

$$\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp\left[a\log\left(\frac{P_{t+h}}{P_t}\right)\right]\mathbb{1}_{\left\{b\log\left(\frac{P_{t+h}}{P_t}\right)<y\right\}}\middle| X_t = x\right).$$

Let us denote the expression above by $g^*(a,b,y,h,x)$. With this notation, the price of a put option (eq. 16) reads:

$$\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}(K-P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right) \\
= \; & K\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\mathbb{1}_{\{r^*_{t+1}+\cdots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\
& -P_t\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp(r^*_{t+1}+\cdots+r^*_{t+h})\mathbb{1}_{\{r^*_{t+1}+\cdots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\
= \; & Kg^*(0,1,\log(K)-\log P_t,h,X_t) - P_t g^*(1,1,\log(K)-\log P_t,h,X_t).
\end{aligned}$$

– Online Appendix –

Disastrous Defaults

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O.1 A general model of X_t's dynamics

We consider the state vector $X_t = [F_t', N_t', N_{t-1}']'$, with $F_t = [x_t, y_t, w_t]'$. Vectors F_t and N_t are, respectively, n_F-dimensional and J-dimensional. Conditional on $\underline{X_t} = \{X_t, X_{t-1}, \dots\}$, the different components of F_{t+1} are independent and drawn from non-centered Gamma distributions:[33]

$$F_{i,t+1}|\underline{X_t} \sim \gamma_{\nu_i}(\zeta_{i,0} + \zeta_{i,F}' F_t + \zeta_{i,n}' n_t, \mu_i),$$

where ν_i, $\zeta_{i,0}$ and μ_i are scalar, $\zeta_{i,F}$ is a n_F-dimensional vector and $\zeta_{i,n}$ is a J-dimensional vector.

In this case, we have (Monfort et al., 2017):

$$\mathbb{E}_t\left(\exp(w'F_{t+1})\right) = \exp\left(a(w)'F_t + b(w)'N_t + c(w)'N_{t-1} + d(w)\right), \text{ for any } w \in V, \quad \text{(a.18)}$$

with

$$\begin{aligned} a(w) &= \zeta_F\left(\frac{w\odot\mu}{1-w\odot\mu}\right), \quad b(w) = \zeta_n\left(\frac{w\odot\mu}{1-w\odot\mu}\right), \quad c(w) = -b(w), \\ d(w) &= \zeta_0'\left(\frac{w\odot\mu}{1-w\odot\mu}\right) - \nu' \log(1-w\odot\mu), \end{aligned} \quad \text{(a.19)}$$

and where V is the set of vector w whose components w_i are in $]-\infty, 1/\mu_i[$, $\zeta_F = [\zeta_{1,F}, \dots, \zeta_{n_F,F}]$, $\zeta_n = [\zeta_{1,n}, \dots, \zeta_{J,n}]$, $\zeta_0 = [\zeta_{1,0}, \dots, \zeta_{n_F,0}]'$, $\mu = [\mu_1, \dots, \mu_{n_F}]'$, $\nu = [\nu_1, \dots, \nu_{n_F}]'$, $\odot$ is the element-by-element (Hadamard) product and where, by abuse of notations, the log and division operator are applied element wise.

[33] The random variable W is drawn from a non-centered Gamma distribution $\gamma_\nu(\varphi, \mu)$, iif there exists a Poisson $\mathscr{P}(\varphi)$-distributed variable Z such that $W|Z \sim \gamma(\nu + Z, \mu)$ where Z and μ are, respectively, the shape and scale parameters of the Gamma distribution (see e.g. Gouriéroux and Jasiak, 2006). When $Z = 0$ and $\nu = 0$, then $W = 0$. When $\nu = 0$, this distribution is called Gamma_0 distribution; this case is introduced and studied by Monfort et al. (2017).

Besides, conditional on $\Omega_t^* = \{F_{t+1}, \Omega_t\}$, we assume that $n_{j,t} = \Delta N_{j,t},\ j = 1,\dots,J$ are independent with Poisson distributions:

$$n_{j,t+1}|\Omega_t^* \sim \mathscr{P}(\beta_j' F_{t+1} + c_j' n_t + \gamma_j). \tag{a.20}$$

Proposition O.1. *The log conditional Laplace transform of process (X_t), denoted by $\psi(v, X_t)$ and defined by:*

$$\mathbb{E}_t\left(\exp(v' X_{t+1})\right) = \exp(\psi(v, X_t)), \tag{a.21}$$

is affine in X_t. That is, we have:

$$\psi(v, X_t) = \psi_0(\nu) + \psi_1(\nu)' X_t, \tag{a.22}$$

with

$$\psi_0(v) = d\left(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A\right) + \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\gamma_j,$$

and where $\psi_1(v) = [\psi_{1,A}(v)', \psi_{1,B}(v)', \psi_{1,C}(v)']'$, where:

$$\begin{cases} \psi_{1,A}(v) &= a(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A), \\ \psi_{1,B}(v) &= b(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A) + \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)c_j + v_B + v_C, \\ \psi_{1,C}(v) &= c(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A) - \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)c_j, \end{cases} \tag{a.23}$$

with $v = [v_A', v_B', v_C']'$, v_A being a n_F-dimensional vector and v_B and v_C being J-dimensional vectors.

Proof. We have:

$$\begin{aligned} & \mathbb{E}_t\left(\exp((v_A', v_B', v_C')X_{t+1})\right) \\ = \;& \mathbb{E}_t\left(\exp(v_A' F_{t+1} + v_B' N_{t+1} + v_C' N_t)\right) = \mathbb{E}_t\left(\mathbb{E}\left(\exp(v_A' F_{t+1} + v_B' N_{t+1} + v_C' N_t)\,\middle|\, \Omega_t, F_{t+1}\right)\right) \\ = \;& \mathbb{E}_t\left(\exp(v_A' F_{t+1} + (v_B + v_C)' N_t)\mathbb{E}\left(\exp(v_B'(N_{t+1} - N_t))\,\middle|\, \Omega_t^*\right)\right) \\ = \;& \mathbb{E}_t\left(\exp(v_A' F_{t+1} + (v_B + v_C)' N_t)\mathbb{E}\left(\exp\left[\sum_{j=1}^{J} v_{B,j}(N_{j,t+1} - N_{j,t})\right]\middle|\, \Omega_t^*\right)\right) \\ = \;& \mathbb{E}_t\left(\exp\left(v_A' F_{t+1} + (v_B + v_C)' N_t + \sum_{j=1}^{J}\left(\beta_j' F_{t+1} + c_j'(N_t - N_{t-1}) + \gamma_j\right)\left(e^{v_{B,j}} - 1\right)\right)\right), \end{aligned}$$

the last equality resulting from the independence of the $n_{j,t+1}$s conditional on Ω_t^*. Therefore:

$$\begin{aligned}
&\mathbb{E}_t\left(\exp((v_A', v_B', v_C')X_{t+1})\right)\\
= \ &\exp\left(\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)c_j + v_B + v_C\right\}' N_t - \left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)c_j\right\}' N_{t-1} + \sum_{j=1}^{J}\gamma_j(e^{v_{B,j}}-1)\right)\times\\
&\mathbb{E}_t\left(\exp\left(\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)\beta_j + v_A\right\}' F_{t+1}\right)\right),
\end{aligned}$$

and the result follows. □

O.2 Conditional and unconditional moments of $[F_t', n_t']'$

We have (see Monfort et al., 2017):

$$\begin{aligned}
\mathbb{E}(F_{t+1}|\underline{X_t}) &= \mu\odot(\zeta_0+\nu)+\mu\odot(\zeta_F' F_t+\zeta_n' n_t) & \text{(a.24)}\\
&=: \mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t,\\
\mathbb{V}ar(F_{t+1}|\underline{X_t}) &= \text{diag}\left[\mu\odot\mu\odot(2\zeta_0+\nu)+2(\{\mu\odot\mu\}\mathbf{1}')\odot(\zeta_F' F_t+\zeta_n' n_t)\right] & \text{(a.25)}\\
&=: \text{diag}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right),
\end{aligned}$$

where $\mathbf{1}$ is a n_F-dimensional vector of ones. Using further eq. (a.20), we obtain:

$$\begin{aligned}
\mathbb{E}_t\left(\begin{bmatrix} F_{t+1}\\ n_{t+1}\end{bmatrix}\right) &= \begin{bmatrix}\mu_F\\ \gamma+\beta'\mu_F\end{bmatrix}+\begin{bmatrix}\Phi_{FF} & \Phi_{Fn}\\ \beta'\Phi_{FF} & \beta'\Phi_{Fn}+c'\end{bmatrix}\begin{bmatrix}F_t\\ n_t\end{bmatrix}\\
\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1}\\ n_{t+1}\end{bmatrix}\right) &= \mathbb{E}_t\left(\mathbb{V}ar\left(\begin{bmatrix} F_{t+1}\\ n_{t+1}\end{bmatrix}\middle|\underline{X_t}^*\right)\right)+\mathbb{V}ar_t\left(\mathbb{E}\left(\begin{bmatrix} F_{t+1}\\ n_{t+1}\end{bmatrix}\middle|\underline{X_t}^*\right)\right)\\
&= \mathbb{E}_t\left(\begin{bmatrix}0 & 0\\ 0 & \text{diag}(\beta' F_{t+1}+c'n_t+\gamma)\end{bmatrix}\right)+\mathbb{V}ar_t\left(\begin{bmatrix}F_{t+1}\\ \beta' F_{t+1}+c'n_t+\gamma\end{bmatrix}\right)\\
&= \begin{bmatrix}0_{n_F\times n_F} & 0_{n_F\times J}\\ 0_{J\times n_F} & \text{diag}(\beta'(\mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t)+c'n_t+\gamma)\end{bmatrix}+\\
&\quad \begin{bmatrix}I_{n_F}\\ \beta'\end{bmatrix}\text{diag}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right)\begin{bmatrix}I_{n_F} & \beta\end{bmatrix}\\
&= \begin{bmatrix}0_{n_F\times J}\\ I_J\end{bmatrix}\text{diag}(\beta'(\mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t)+c'n_t+\gamma)\begin{bmatrix}0_{J\times n_F} & I_J\end{bmatrix}+\\
&\quad \begin{bmatrix}I_{n_F}\\ \beta'\end{bmatrix}\text{diag}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right)\begin{bmatrix}I_{n_F} & \beta\end{bmatrix}.
\end{aligned}$$

Using the fact that, for any n-dimensional vector a:

$$vec(diag(a)) = \sum_{i=1}^{n} vec(diag[e_i e_i' a]) = \sum_{i=1}^{n} vec(e_i e_i') a_i = \underbrace{\left(\sum_{i=1}^{n} vec(e_i e_i') e_i' \right)}_{=:S_n} a,$$

we obtain:

$$\begin{aligned}
& vec\left(\mathbb{V}ar_t \left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array} \right] \right) \right) \\
= \quad & \left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \right) S_J(\beta'(\mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t) + c' n_t + \gamma) + \\
& \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \right) S_{n_F} \left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t \right) \\
= \quad & \left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \right) S_J(\beta' \mu_F + \gamma) + \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \right) S_{n_F} \mu_F^{var} + \\
& \left\{ \left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \right) S_J \beta' \Phi_{FF} + \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \right) S_{n_F} \Phi_{FF}^{var} \right\} F_t + \\
& \left\{ \left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array} \right] \right) S_J (\beta' \Phi_{Fn} + c') + \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array} \right] \right) S_{n_F} \Phi_{Fn}^{var} \right\} n_t.
\end{aligned}$$

Therefore:

$$\begin{aligned}
& vec\left(\mathbb{V}ar_t \left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array} \right] \right) \right) \\
= \quad & \left[\begin{array}{c}
S_{n_F} \left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t \right) \\
(I_{n_F} \otimes \beta') S_{n_F} \left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t \right) \\
(\beta' \otimes I_{n_F}) S_{n_F} \left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t \right) \\
S_J(\beta'(\mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t) + c' n_t + \gamma) + (\beta' \otimes \beta') S_{n_F} \left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t \right)
\end{array} \right]
\end{aligned}$$

$$= \begin{bmatrix} S_{n_F}\mu_F^{var} \\ (I_{n_F}\otimes\beta')S_{n_F}\mu_F^{var} \\ (\beta'\otimes I_{n_F})S_{n_F}\mu_F^{var} \\ S_J(\beta'\mu_F+\gamma)+(\beta'\otimes\beta')S_{n_F}\mu_F^{var} \end{bmatrix} +$$

$$\begin{bmatrix} S_{n_F}\Phi_{FF}^{var} \\ (I_{n_F}\otimes\beta')S_{n_F}\Phi_{FF}^{var} \\ (\beta'\otimes I_{n_F})S_{n_F}\Phi_{FF}^{var} \\ S_J\beta'\Phi_{FF}+(\beta'\otimes\beta')S_{n_F}\Phi_{FF}^{var} \end{bmatrix} F_t + \begin{bmatrix} S_{n_F}\Phi_{Fn}^{var} \\ (I_{n_F}\otimes\beta')S_{n_F}\Phi_{Fn}^{var} \\ (\beta'\otimes I_{n_F})S_{n_F}\Phi_{Fn}^{var} \\ S_J\beta'\Phi_{Fn}+(\beta'\otimes\beta')S_{n_F}\Phi_{Fn}^{var} \end{bmatrix} n_t.$$

Hence, both $vec\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)\right)$ and $\mathbb{E}_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)$ are affine functions of $\begin{bmatrix} F_t \\ n_t \end{bmatrix}$. Let us introduce the following notations:

$$\begin{aligned} \mathbb{E}_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) &= \mu_E + \Phi_E \begin{bmatrix} F_t \\ n_t \end{bmatrix} \\ vec\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)\right) &= \mu_V + \Phi_V \begin{bmatrix} F_t \\ n_t \end{bmatrix}. \end{aligned}$$

Assuming that $\begin{bmatrix} F_t \\ n_t \end{bmatrix}$ is covariance-stationary, we have:

$$\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \mathbb{E}\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) = \mu_E + \Phi_E \mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right),$$

which leads to:

$$\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = (I-\Phi_E)^{-1}\mu_E.$$

We also have:

$$\mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \mathbb{E}\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) + \mathbb{V}ar\left(\mathbb{E}_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right).$$

Using that:

$$\begin{cases} \mathbb{E}\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = \mathbb{E}\left(\mu_V + \Phi_V \begin{bmatrix} F_t \\ n_t \end{bmatrix}\right), \\ \mathbb{V}ar\left(\mathbb{E}_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = \mathbb{V}ar\left(\mu_E + \Phi_E \begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \Phi_E \mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\Phi_E', \end{cases}$$

we get:

$$vec\left(\mathbb{V}ar\left(\left[\begin{array}{c}F_t\\ n_t\end{array}\right]\right)\right)=(I-\Phi_E\otimes\Phi_E)^{-1}\left[\mu_V+\Phi_V\mathbb{E}\left(\left[\begin{array}{c}F_t\\ n_t\end{array}\right]\right)\right].$$

O.3 Proof of Lemma 1

We have:

$$\begin{aligned}
r^s_{t+1} &= \log\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right)\\
&= \log(P_{t+1}+D_{t+1})-\log(P_t)+\log(P_{t+1})-\log(P_{t+1})+\\
&\quad \log(D_{t+1})-\log(D_{t+1})+\log(D_t)-\log(D_t)\\
&= \tau_{t+1}-\tau_t+g_{d,t+1}+\log\left(1+\frac{D_{t+1}}{P_{t+1}}\right). \qquad \text{(a.26)}
\end{aligned}$$

Besides:

$$\begin{aligned}
\log[1+D_{t+1}/P_{t+1}] &= \log[1+\exp(-\tau_{t+1})]\\
&\approx \log[1+\exp(-\bar{\tau})\{1-(\tau_{t+1}-\bar{\tau})\}]\\
&\approx \log[1+\exp(-\bar{\tau})-\exp(-\bar{\tau})(\tau_{t+1}-\bar{\tau})]\\
&\approx \log[1+\exp(-\bar{\tau})]+\log\left[1-\frac{\exp(-\bar{\tau})(\tau_{t+1}-\bar{\tau})}{1+\exp(-\bar{\tau})}\right]\\
&\approx \log[1+\exp(-\bar{\tau})]-\frac{\tau_{t+1}-\bar{\tau}}{1+\exp(\bar{\tau})}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
r^s_{t+1} &\approx \tau_{t+1}-\tau_t+g_{d,t+1}+\log[1+\exp(-\bar{\tau})]-\frac{\tau_{t+1}-\bar{\tau}}{1+\exp(\bar{\tau})}\\
&\approx \log[1+\exp(-\bar{\tau})]+\frac{\bar{\tau}}{1+\exp(\bar{\tau})}+\frac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})}\tau_{t+1}-\tau_t+g_{d,t+1}\\
&\approx \log[1+\exp(\bar{\tau})]-\frac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})}\bar{\tau}+\frac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})}\tau_{t+1}-\tau_t+g_{d,t+1},
\end{aligned}$$

which leads to eq. (a.14).

O.4 Credit Default Swap

The Credit Default Swap (CDS) is the most common credit derivative. It is an agreement between a protection buyer and a protection seller, whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit event, such as bankruptcy or failure to pay, of a reference entity. The contingent payment usually replicates the loss incurred by a creditor of the reference entity in the event of its default (see e.g. Duffie, 1999).

More specifically, a CDS works as follows: the protection buyer pays a regular (annual, semi-annual or quarterly) premium to the so-called protection seller. These payments end either after a given period of time (the maturity of the CDS), or at default of the reference entity i from segment j. In the case of default of this debtor, the protection seller compensates the protection buyer for the loss the latter would incur upon default of the reference entity (assuming that the latter effectively holds a bond issued by the reference entity). The CDS spread, also called CDS premium, is the regular payment paid by the protection buyer (expressed in percentage of the notional and in annualized terms). Since, in our model, the segments of credit are homogeneous, the CDS spreads are the same for all entities belonging to the same segment. Let us denote by $S^{CDS}_{j,t,h}$ the maturity-h CDS spread of segment-j entities, by q the number of premium payments made per year and by RR the recovery rate.[34]

Let $d_{j,i,t}$ be the indicator of default of entity i belonging to segment j: $d_{j,i,t} = 1$, if entity i is in default at time t (or before) and $d_{j,i,t} = 0$, otherwise.[35] Note that we have $N_{j,t} = \Sigma_{i=1}^{I_j} d_{j,i,t}$.

At inception of the CDS contract, there is no cash-flow exchanged between both parties and the CDS spread $S^{CDS}_{j,t,h}$ is determined so as to equalize the present discounted values of the payments promised by each of them. If the maturity h is expressed in years, we have:

$$\underbrace{\mathbb{E}^{\mathbb{Q}}_t \left\{ \sum_{k=1}^{qh} \Lambda_{t,t+k}(1-RR)(d_{j,i,t+k} - d_{j,i,t+k-1}) \right\}}_{\text{Protection leg}} = \underbrace{\mathbb{E}^{\mathbb{Q}}_t \left\{ \frac{S^{CDS}_{j,t,h}}{q} \sum_{k=1}^{qh} \Lambda_{t,t+k}(1-d_{j,i,t+k}) \right\}}_{\text{Premium leg}}. \quad \text{(a.27)}$$

By expanding the latter equality, the CDS spread $S^{CDS}_{j,t,h}$ is easily derived if one can compute $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k})$, $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} d_{j,i,t+k})$ and $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} d_{j,i,t+k-1})$ for all $k > 0$. By symmetry arguments, assuming that $d_{j,i,t} = 0$, we have:

$$\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} d_{j,i,t+k}) = \mathbb{E}^{\mathbb{Q}}_t\left(\Lambda_{t,t+k} \frac{\overline{N_{j,t+k}} - \overline{N_{j,t}}}{I_j - \overline{N_{j,t}}}\right) = \frac{1}{I_j - \overline{N_{j,t}}}\left(\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k}\overline{N_{j,t+k}}) - \overline{N_{j,t}}\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k})\right).$$

where $\overline{N_{j,t}} = \min(N_{j,t}, I_j)$. While exact formulas are available to compute these quantities, we will proceed under the assumption that the probability of having $N_{j,t} > I_j$ is so small that we have, in

[34] While the model can be extended to accommodate stochastic recovery rates, we restrict our attention to deterministic recovery rates as is common practice in pricing exotic credit derivatives.

[35] In what follows, we augment the filtration Ω_t with $\underline{d_t}$.

particular, $\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}\overline{N_{j,t+k}}) \approx \mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{j,t+k})$. In this context, we obtain:

$$\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}d_{j,i,t+k}) \approx \frac{1}{I_j - N_{j,t}} \left(\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{j,t+k}) - N_{j,t}\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k})\right). \tag{a.28}$$

Similarly, we obtain:

$$\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}d_{j,i,t+k-1}) \approx \frac{1}{I_j - N_{j,t}} \left(\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{j,t+k-1}) - N_{j,t}\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k})\right). \tag{a.29}$$

Using eqs. (a.28) and (a.29) in eq. (a.27), we get:

$$S_{j,t,h}^{CDS} \approx q(1-RR)\frac{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}[N_{j,t+k}-N_{j,t+k-1}]\right\}}{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}(I_j - N_{j,t+k})\right\}}.$$

This expression is the same as that for spreads of credit indexes (eq. 9).

O.5 Maximum Sharpe ratio between dates t and $t+h$

The maximum Sharpe ratio of an investment realized between dates t to $t+h$ is given by (see Hansen and Jagannathan, 1991):

$$\mathscr{M}_{t,t+h} = \frac{\sqrt{\mathbb{V}ar_t(M_{t,t+h})}}{\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+h})}.$$

Using the notation $M_{t,t+1} = \exp(\mu_{0,m} + \mu_{1,m}'X_{t+1} + \mu_{2,m}'X_t)$, we have:

$$\begin{aligned} M_{t,t+h} &= \exp\left(h\mu_{0,m} + \mu_{2,m}'X_t\right) \times \\ &\quad \exp\left([\mu_{1,m}+\mu_{2,m}]'X_{t+1} + \dots + [\mu_{1,m}+\mu_{2,m}]'X_{t+h-1} + \mu_{1,m}'X_{t+h}\right) \end{aligned}$$

Therefore:

$$\mathscr{M}_{t,t+h} = \frac{\sqrt{\theta_{t,h}(2[\mu_{1,m}+\mu_{2,m}], 2\mu_{1,m}) - \theta_{t,h}(\mu_{1,m}+\mu_{2,m}, \mu_{1,m})^2}}{\theta_{t,h}(\mu_{1,m}+\mu_{2,m}, \mu_{1,m})},$$

where

$$\theta_{t,h}(u,v) = \mathbb{E}_t(\exp(u'X_{t+1} + \dots + u'X_{t+h-1} + v'X_{t+h})).$$

When X_t is an affine process, $\theta_{t,h}(u,v)$ can be computed in closed-form by using recursive formulas as in Prop. 5 under the physical measure instead of the risk-neutral one and setting the risk-free short-term rate to 0, i.e. by using $\eta_0 = 0$ and $\eta_1 = 0$.