Public Liquidity Demand and Central Bank Independence*

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November 4, 2019

Abstract

This paper studies how private demand for public liquidity affects the independence of a central bank vis-à-vis the fiscal authority. Whereas supplying liquidity to the private sector creates degrees of freedom for both fiscal and monetary authorities, we show that the authority that is most able to attract private liquidity demand can ultimately impose its views to the other.

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1 Introduction

A central bank is independent if the fiscal authority does not stand in the way of its objectives, foremost among them being price stability. In workhorse macroeconomic models, a necessary condition for equilibrium, the "intertemporal budget constraint of the government", imposes strong restrictions on jointly feasible fiscal and monetary policies. In particular, the central bank is independent only if the fiscal authority can commit to a Ricardian policy, ensuring that the budget constraint holds for all paths of the price level.[1]

This paper extends these workhorse models in two directions. We first posit that the public sector has a unique ability to supply liquidity vehicles to the economy, thereby generating resources above and beyond fiscal surpluses. This may relax the interdependence between fiscal and monetary policies that derives from a standard intertemporal budget constraint. Second, rather than assuming that fiscal and monetary authorities indefinitely commit to policy rules, we endow both authorities with objectives and instruments, and study the subgame-perfect outcome from their strategic interactions. Put simply, we offer a formal game-theoretic analysis of Wallace's "game of chicken".

One purpose of these extensions is to assess central-bank independence in the current context. Several observers (e.g., Blanchard, 2019) argue that the US government should reap the benefits from interest rates below growth rates by issuing more debt at zero fiscal and inflationary costs. Current attempts of the US executive branch at influencing monetary policy are apparently not perceived by markets as pure noise Bianchi et al. (2019). Overall, relative to a view of the world in which central-bank independence is warranted by a Ricardian fiscal policy given the intertemporal budget constraint of the government, the current context suggests on one hand that the budget constraint may be "soft", but that fiscal policy, on the other hand, is far from Ricardian. What is the net implication for central-bank independence?

Our analysis generates the following insights. First, we offer a general formulation of the condition under which public liquidity supply makes the monetary arithmetic "pleasant", in the sense that it relaxes fiscal and monetary interdependence and thus significantly expands the set of jointly feasible policies. The condition is that the public sector must be able to indefinitely rollover securities that are not backed by any fiscal surplus, and that it must be able to do so at a sufficiently low cost relative to the return on pure consumption claims. A simple but noteworthy insight is that it suffices that only one type of public liabilities, e.g., central-bank reserves, can be rolled over at such a low cost for the monetary arithmetic to be pleasant even if other liabilities, such as

[1] For the institutional aspect of central bank independence, see Cukierman (2008).

government bonds, carry a higher yield. Under such pleasant monetary arithmetic, the central bank retains significant degrees of freedom in determining the price level even when fiscal policy is not Ricardian.

One could conclude from this latter remark that demand for public liquidity reinforces central-bank independence. The predictions from our strategic analysis are however gloomier. We find that the degree of "pleasantness" of the economy, broadly defined as the wiggle room between fiscal and monetary policies, is actually not the essential determinant of central-bank independence when both authorities are strategic. It is rather its ability to mop up private liquidity demand before the fiscal authority does so with the issuance of debt that warrants the independence of the central bank. We find indeed that the authority that is the fastest at meeting private liquidity demand can force the other to chicken out. There is fiscal consolidation and a stable price level if the monetary authority preempts liquidity demand whereas there is fiscal expansion and inflation in case the fiscal authority does so.

Overall, our explicit strategic approach offers useful insights into the question that Sargent and Wallace (1981) raise in conclusion of their unpleasant arithmetic: "The question is, Which authority moves first, the monetary authority or the fiscal authority? In other words, Who imposes discipline on whom?" We contend that the authority who moves first is the one that preempts private demand for liquidity. In times in which liquidity demand is high and the central bank has the exclusive ability to satisfy it, then it is de facto independent. When bonds and reserves are substitutes, the fiscal authority may by contrast preempt the proceeds from supplying liquidity and spend them, thereby stretching public finances and forcing the central bank to inflate away public debt.

The paper is organized as follows. For expositional clarity, we derive our main insights in a very simple overlapping-generations model in which informational asymmetries in the credit market create room for valuable public liquidity supply. Section 2 derives the implications of a pleasant monetary arithmetic in this simple framework, and Section 3 solves for Wallace's game of chicken in it. Section 4 is more abstract in nature. It offers general and yet compact conditions under which monetary arithmetic is pleasant in a broad class of models. It aims in particular at confirming that our main results do not live or die on the overlapping-generations structure used to illustrate them.

Literature review. This paper is connected to the literature on the fiscal theory of the price level starting with Leeper (1991), Sims (1994) and Woodford (1994, 1995, 2001), and to its criticism (see Buiter, 2002; McCallum, 2001; Niepelt, 2004, among others). It relates in particular to the literature investigating whether the fiscal theory of the price level applies in non-Ricardian environments, starting with Bénassy (2008). More recent contributions include Bassetto and Cui (2018) and Farmer and Zabczyk (2019). Our characterization of a pleasant monetary arithmetic builds in particular on Bassetto and Cui (2018), who show that low interest rates on public debt prevents fiscal policy from selecting a unique price level. Our main contribution relative to these papers is to go beyond the determination of the set of feasible policies and predict the ones that actually arise given the strategic interactions between fiscal and monetary authorities.

This paper is connected with the literature on bubbles (e.g. Tirole, 1985, among many others) and, in particular, with models of bubbles where the economy is dynamically efficient as in Farhi and Tirole (2012) or Martin and Ventura (2012). From this perspective, our paper is connected to the literature linking monetary policy to bubbles including Gali (2014). In particular, Asriyan et al. (2019) consider the competition between private bubbles and a public one ("money"). In contrast, we consider the competition between different public bubbles such as money or reserves and government bonds.

The idea that public debt is used as private liquidity goes back to at least Diamond (1965) and was widely studied since them (see Woodford, 1990; Holmström and Tirole, 1998, among others). Krishnamurthy and Vissing-Jorgensen (2012) showed in the data that public debt shared many of the properties of money.

Finally, this paper is also connected to the literature on central bank independence and central bank's balance sheet (Sims, 2003; Hall and Reis, 2015; Del Negro and Sims, 2015). In contrast with this literature, we show that the central bank can loose its independence not when it has to be recapitalized by the government, when in the opposite situation where the central bank is forced to make transfers to the government.

2 Pleasant monetary arithmetic

This section shows how the demand for public liquidity relaxes the interdependence of monetary and fiscal policies. To this purpose, we introduce an overlapping generation model of entrepreneurs. We first study how liquidity demand shapes the set of price levels that a central bank can target for a given fiscal policy. Second, we extend our analysis to the presence of multiple public liabilities. In all these situations, we show that liquidity demand expands the set of price levels that the central bank can target.

2.1 Setup

Let us consider an overlapping-generations model in which the trading of public liabilities help overcome informational frictions in the credit market.

Time is discrete and indexed by $t \in \mathbb{N}$. There is a single consumption good. The economy is populated by a public sector and two types of private agents, savers and entrepreneurs. At each date, a unit mass of entrepreneurs and a unit mass of savers are born. They live for two dates and value consumption only when old, at which time they are risk-neutral. All agents use the same currency as a unit of account.

Savers. Young date-t savers receive a real endowment that they can store with a linear return $e^{-\delta}$. Endowments are i.i.d. across savers of a given cohort. Their distribution has a mean $1+\bar{\tau}_t$, where $\bar{\tau}_t \geq 0$ is the lower bound of its support. Without loss of generality, the minimum endowment $\bar{\tau}_t$ is strictly positive at date 1, zero otherwise.

Entrepreneurs. Young date-t entrepreneurs are endowed with a storage technology with a random linear return. The (gross) return has expected value e^{ρ}, where $\rho > \max\{-\delta; 0\}$, and its distribution has 0 in its support. Returns are perfectly correlated across entrepreneurs of the same cohort. Entrepreneurs are competitive on the credit market.

Public sector. The public sector sets transfers to the private sector and sets the price levels.[2] It starts out with an exogenous legacy nominal liability $L > 0$ due at date 1. We generalize to a legacy nominal liability with multiple maturities in Appendix [TBD]. It can issue one-period nominal bonds. Let $D_t \geq 0$ denote the number of currency units due at date $t+1$ and promised at date t.[3] The date-t real price of debt is $\phi_t \geq 0$. We denote by σ_t and τ_t the transfers from the young entrepreneurs and the young savers respectively to the public sector. The date-t real fiscal surplus, denoted by s_t, is simply $s_t = \sigma_t + \tau_t$.

One possible interpretation of this setting is that the public sector has issued long-term debt in an unmodelled past (before date 0) and that L is the residual amount due at date 1. An alternative interpretation is that of a bailout decision following a major financial crisis. Under this interpretation, the liabilities L are that of a distressed (unmodelled) financial sector, and "default" therefore corresponds to an incomplete bailout.

Information structure. The public sector does not observe savers' endowments, entrepreneurs' realized returns, nor trades by private agents. There exists $T \in \mathbb{N}$ such that if t does not belong to $\{1+k(T+1), k \geq -1\}$, then savers born at date t perfectly observe

[2] We deliberately posit that the monetary authority can implement whichever price level it wants without explicitly modelling a particular implementation. We believe that this is the right benchmark to start with when studying an explicit model of strategic interactions between fiscal and monetary authorities.

[3] The exclusion of public savings could be replaced by a no-Ponzi game condition without adding new insights.

the return realized by date-t entrepreneurs at date $t+1$. Otherwise, they do not observe it.

That $\rho > -\delta$ implies that (risky) loans from savers to entrepreneurs unlock gains from trades. Such a private credit market works seamlessly for the cohorts that do not experience any informational asymmetries between lenders and borrowers. At dates that belong to $\{1 + k(T+1), k \geq -1\}$, however, the credit market collapses as entrepreneurs can always claim at the next date that their realized return is zero. Thus they cannot pledge any future output to savers. Accordingly, we interpret the dates at which there are no informational asymmetries between savers and entrepreneurs as "normal times," and the ones in which the credit market shuts down as "financial crises."

Notice that the case $T = 0$ corresponds to an environment in which savers never observe entrepreneurs' realized returns. That is a low real rate environment, where the public sector can raise debt at a gross real interest rate $e^{-\rho}$. This environment is very similar to dynamic inefficient ones studied for instance by Bassetto and Cui (2018).

2.2 Feasible policies

We define a feasible policy as a policy sequence $(D, \sigma, \tau, P)_{t\in\mathbb{N}}$ satisfying (i) solvency of the public sector; (ii) market clearing; (iii) savers and entrepreneurs optimize; (iv) savers are indifferent between public and private investment.

The last feasibility condition (iv) imposes that $\phi_t = e^{\delta}$ when t belongs to $\{1 + k(T+1), k \geq -1\}$, $\phi_t = e^{-\rho}$ otherwise.

For all $t \in \mathbb{N}$, the budget constraint of the public sector reads

$$\mathbb{1}_{\{t=1\}} \frac{L}{P_t} + \frac{D_{t-1}}{P_t} = \sigma_t + \tau_t + \phi_t \frac{D_t}{P_{t+1}}, \tag{1}$$

where $D_{-1} = 0$ by convention and the dummy variable $\mathbb{1}_{\{t=1\}}$ equals 1 when $t = 1$ and 0 otherwise.

Note first that the public sector cannot raise taxes except at date 1: Given the distribution of endowments and returns, savers and entrepreneurs can always claim that they are penniless so as to avoid taxation. This implies in turn that for all $t > 1$,

$$\phi_t \frac{D_t}{P_{t+1}} \geq \frac{D_{t-1}}{P_t}. \tag{2}$$

Unpleasant monetary arithmetic Let us first focus on the case where financial crises are not frequent, that is $T > \delta/\rho$. As a result, $e^{-\delta} (e^{\rho})^T > 1$ and so the public sector cannot issue debt –except at date 0– because any (real) amount to be refinanced would ultimately exceed savers' unit aggregate endowment. Hence, $D_1 = 0$ and the date-0 and date-1 budget constraints result in:

$$\frac{L}{P_1} = \frac{\sigma_0}{\phi_0} + \sigma_1 + \tau_1, \tag{3}$$

where $\sigma_0 \leq 0$ and $\sigma_1 \leq 0$ are transfers to young entrepreneurs and $\tau_1 \in (0, \bar{\tau}_1]$ is a tax on young savers at date 1. In this case, the monetary arithmetic is *unpleasant* in the sense that any reduction in fiscal surpluses at dates 0 and 1 should be associated with a higher price level P_1 (see Section 4 for a more formal definition).

Of particular interest to us is the fact that every feasible fiscal policy $(\sigma_0, \sigma_1, \tau_1)$ *is associated with a unique feasible initial price level* P_1 *determined by the budget constraint of the public sector* (3).

Pleasant monetary arithmetic Suppose now that $T \leq \delta/\rho$. Then, given a fiscal policy $(\sigma_0, \sigma_1, \tau_1)$, there exists a bubble with date-1 real value $\omega \in [0, 1]$. To see this, notice that any such a bubble can be perpetually refinanced since $(e^{\rho})^T e^{-\delta} \leq 1$ and the size of the bubble never exceeds disposable savings.[4] For each $\omega \in [0, 1]$ there exists a unique price level P_1:

$$\frac{L}{P_1} = \frac{\sigma_0}{\phi_0} + \sigma_1 + \tau_1 + \omega. \tag{4}$$

In this case, the monetary arithmetic is *pleasant*, in the sense that reductions in fiscal surpluses at dates 0 and 1 (σ_0 and σ_1) can be financed by an increase in the bubble ω and not necessarily by an adjustment in the price level P_1.

Of particular interest to us is the fact that every feasible fiscal policy $(\sigma_0, \sigma_1, \tau_1)$ *is associated multiple feasible initial price level* P_1.

To wrap up, if $T > \delta/\rho$, then fiscal and monetary policies are strongly interdependent in the sense that each feasible fiscal policy is associated with a unique initial price level. As in the fiscal theory of the price level, real fiscal surpluses dictate the price level. In particular, if a feasible fiscal policy features a lower surplus s_0 (or s_1) than another one, then it must also come at a higher price level. Otherwise, fiscal and monetary policies are less interdependent in the sense that a given feasible fiscal policy is associated with an interval of feasible initial price levels.

[4] The bubble can be described using the public debt. In such a case, $\omega = \phi_1 D_1 / P_2$. For an extended discussion of the feasibility of such a bubble, see section 4.1.

2.3 Feasible policies with multiple public liabilities

This subsection extends the analysis of feasible policies to the case where the public sector may issue two types of liabilities, government debt and central-bank reserves and where the central bank and the government have separate budget constraint as in Bassetto and Messer (2013) or Hall and Reis (2015).

Now the public sector is comprised of two authorities, a fiscal and a monetary authorities. Both authorities can issue debt: Fiscal authority issues government bond denoted by $B_t \geq 0$ and monetary authority issues central-bank reserves denoted by $X_t \geq 0$. Monetary authority cannot operate transfers to the private sector but both authorities can buyback a fraction of the exogenous nominal legacy liability L at date 0 at the prevailing price ϕ_0. We denote by L_M and L_F the debt buyback by the monetary and the fiscal authority respectively. We denote by d_t the transfer from the monetary authority to the fiscal one. This transfer can be either positive (dividend) or negative (recapitalisation).

Budget constraints The budget constraints of the two authorities at dates 0 and 1 write:

$$0 = -d_0 + \phi_0 \left(\frac{X_0}{P_1} - \frac{L_M}{P_1} \right), \tag{5}$$

$$\frac{X_0}{P_1} = -d_1 + \frac{L_M}{P_1} + \phi_1 \frac{X_1}{P_2}, \tag{6}$$

$$0 = d_0 + \sigma_0 + \phi_0 \left(\frac{B_0}{P_1} - \frac{L_F}{P_1} \right), \tag{7}$$

$$\frac{B_0}{P_1} + \frac{L}{P_1} = d_1 + \tau_1 + \sigma_1 + \frac{L_F}{P_1} + \phi_1 \frac{B_1}{P_2}, \tag{8}$$

$$\tag{9}$$

Overall, the consolidated budget constraint is thus similar to equation (4):

$$\frac{L}{P_1} = \frac{\sigma_0}{\phi_0} + \sigma_1 + \tau_1 + \phi_1 \left(\frac{B_1}{P_2} + \frac{X_1}{P_2} \right). \tag{10}$$

When $T > \delta/\rho$, then the last two terms are equal to zero and fiscal policy $(\sigma_0, \sigma_1, \tau_1)$ pins down a unique price level. Otherwise, for a given fiscal policy $(\sigma_0, \sigma_1, \tau_1)$ and a given date-1 real public debt $b_1 = \phi_1 B_1 / P_2$ (unbacked by future real surpluses), there exists a continuum of date-1 real value of central-bank reserves $x_1 =\in [0, 1 - b_1]$. For each level of x_1, there exists a unique price level P_1 such that:

$$\frac{L}{P_1} = \frac{\sigma_0}{\phi_0} + \sigma_1 + \tau_1 + b_1 + x_1. \tag{11}$$

With two liabilities, for a given fiscal policy and for a given bubble attracted by the fiscal authority, the feasible set of price levels is reduced to a singleton when the monetary arithmetic is unpleasant and larger when the arithmetic is pleasant. In the latter case, the larger the bubble attracted by the fiscal authority, the smaller the monetary policy space.

With multiple maturities We extend in Appendix [TBD] these results to multiple maturities of legacy debt. In such a context, Cochrane (2001) shows that the issuance policy of public liability at date 0 matters for the exact price level path. We extend his result and show that when the central bank can issue remunerated reserves, the central bank can pin down the price level at dates 0 and 1.

The analysis thus far has focussed on determining how the "pleasantness" of the monetary arithmetic shapes the set of feasible prices given fiscal policy. Section 2.3 in particular has shown that when both branches of government supply liquidity vehicles to the private sector, then not only fiscal surpluses, but also the real resources stemming from liquidity yields on bonds and reserves, contribute to pin down the price level. These additional resources expand the sets of feasible policies.

It would be incorrect to infer that such an expansion automatically reinforces central-bank independence. Only a model that compares interactions between fiscal and monetary authorities under varying degrees of "pleasantness" can deliver such a prediction.

3 Wallace's game of chicken

This section offers such an explicit model of Wallace's "game of chicken" to investigate how the degree of pleasantness affects the interactions between fiscal and monetary authorities.

In this model, the fiscal authority has a bias towards spending whereas the monetary one has one towards price stability. The private sector has a demand for public assets, both in the form of reserves or government bonds. We stack the deck in favor of central-bank independence by assuming that the monetary authority "moves first" and imposes a price level at the beginning of each period. We show yet that despite being at a disadvantage, the fiscal authority can impose its policy views if it preempts a sufficiently large fraction of public liquidity demand.

3.1 Setup

This section adds strategic interactions to the OLG example described in subsection 2.3. We first specify the actions of the central bank and the government and the timing and, then, we describe the preferences of the two authorities.

Actions and timing. At each date $t \geq 0$,

- M first sets the date-t price level P_t.
- F taxes young savers an amount τ_t up to $\bar{\tau}_t$.
- Savers issue (nominal) demands $\bar{B}_t$ and $\bar{X}_t$ for claims issued by F and M respectively, where $\bar{B}_t, \bar{X}_t \geq 0$.[5] We adopt the convention that $\bar{B}_{-1} = \bar{X}_{-1} = 0$.
- F decides on a supply $B_t \in [0, \bar{B}_t]$ and M on a supply $X_t \in [0, \bar{X}_t]$ and they collect their respective proceeds $\phi_t B_t / P_{t+1}$ and $\phi_t X_t / P_{t+1}$.
- One authority makes a take-it-or-leave-it offer to the other that consists in a real transfer $-\sigma_t$ to young date-t entrepreneurs and a reimbursement to the holders of current liabilities. Current liabilities are the endogenous ones B_{t-1} and X_{t-1} at all dates and the exogenous one L at date 1. At date-0, the offer also includes the full or partial buyback of the exogenous ability L, either by the central bank (L_M) or by the fiscal authority (L_F). If the other authority turns down the offer then each authority uses its proceeds as it sees fit.

Note that, which authority does the take-it-or-leave-it offer is immaterial. Furthermore, when the authorities decide on their supplies of assets and when the offer is turned down, whether authorities move simultaneously or in a particular sequence is also immaterial. This implies that, except for the price level that is set by the central bank followed by the choice of taxes by the fiscal authority, the timing of actions by the central bank and the government is not critical for our results.

Preferences. Denoting $q_{t,t'}$ the real rate of return for savers between t and t'. The respective date-t objectives of F and M are:

$$U_t^F = -\sum_{t' \geq t} q_{t,t'}(\sigma_{t'} - \alpha_F \Delta_{t'}), \tag{12}$$

$$U_t^M = -\sum_{t' \geq t} q_{t,t'}(\mid P_{t'} - P_M \mid + \alpha_M \Delta_{t'}), \tag{13}$$

[5] Whereas such demands are expressed in nominal or real terms is immaterial in our flexible price, perfect-foresight environment.

where $\alpha_F, \alpha_M, P_M > 0$. The variable Δ_t is equal to 1 in case of an outright default of the public sector on any of its claims held by the private sector due at date t, and to 0 otherwise.

In words, each authority $X \in \{F; M\}$ incurs a cost α_X if the public sector nominally defaults. The fiscal authority also incurs a cost from devoting public resources to the repayment of these liabilities. It prefers to spend these resources with transfers to young entrepreneurs. The monetary authority finds it costly to deviate from a given target P_M for the price level. To lift equilibrium indeterminacy, we also assume that M, when indifferent among several actions, prefers the ones that maximize transfers to entrepreneurs.

Why a separate central bank? For brevity, we simply posit that the public sector is comprised of two distinct authorities with different objective functions. Yet a simple time-inconsistency argument could micro-found the delegation of price-level determination to a monetary authority. Suppose that the social welfare function puts more weight on entrepreneurs than on savers, but that the government lacks commitment. In this case, if private agents use nominal contracts, such a government would be tempted to inflate away old entrepreneurs' debts ex-post so as to transfer consumption to them from old savers. Savers would anticipate this, and this would inefficiently shut down credit markets ex-ante. Delegation to an entity with a mandate for a stable price level solves this problem. Our setting is one in which this entity cannot fully commit to a path of price levels, however, because it also cares about default. Another way of saying this is that the case in which $\alpha_M = 0$ is that in which M can fully and credibly commit to set the price level to P_M, as setting $P_t = P_M$ at all dates clearly is a dominant strategy in this case.

3.2 Equilibrium concept

Let us introduce the definition of an equilibrium in our setting. This definition should ensure that the sequence of private demands for public liabilities ($\bar{B}_t$ and $\bar{X}_t$) solve the savers' problem, given the policies decided by the fiscal and the monetary authorities, and, given these demands, the two authorities play a subgame perfect Nash equilibrium. More precisely:

Definition 1. ***(Equilibrium)*** *An equilibrium is a policy sequence $(B_t, X_t, P_t, \sigma_t, \tau_t)_{t\geq 0}$ and date-0 buybacks (L_F, L_M), a sequence of liquidity demand $(\bar{B}_t, \bar{X}_t)_{t\geq 0}$, and a sequence of price $(\phi_t)_{t\geq 0}$ such that all agents have perfect foresight and for all $t \geq 0$,*

- *Given a policy sequence and a sequence of price, the demand for liquidity is optimal:*

$$\phi_t \frac{\bar{X}_t + \bar{B}_t}{P_{t+1}} \leq 1, \tag{14}$$

$$\mathbb{1}_{\{t=1\}} \frac{L - L_F - L_M}{P_t} + \frac{\bar{X}_{t-1} + \bar{B}_{t-1}}{P_t} \leq -\mathbb{1}_{\{t=0\}} \phi_t \frac{L_F + L_M}{P_{t+1}} + \phi_t \frac{X_t + B_t}{P_{t+1}} + \sigma_t + \tau_t, \tag{15}$$

and the price of public liabilities is such that savers are indifferent between public and private investment:

$$\phi_t = e^{\delta} \text{ if } t \in \{1 + k(T+1), k \geq -1\},\ \phi_t = e^{-\rho} \text{ otherwise.} \tag{16}$$

- *Given the demand for liquidity $(\bar{B}_t, \bar{X}_t)_{t\geq 0}$ and a sequence of price $(\phi_t)_{t\geq 0}$, the date-t continuation of the policy sequence consitutes a subgame perfect Nash equilibrium between F and M and satisfy:*

$$\sigma_t \leq 0 \text{ and } \tau_t \leq \bar{\tau}_t, \tag{17}$$

$$B_t \leq \bar{B}_t \text{ and } X_t \leq \bar{X}_t \tag{18}$$

$$L_F + L_M \leq L \tag{19}$$

Condition (14) ensures that savers' total unit endowment suffices to fund their real demands for the public sector's liabilities, condition (15) imposes that savers rationally anticipate default along the equilibrium path. Finally, prices are such that savers are indifferent between public and private investments imposing condition (16).

Inequality (17) reflects that entrepreneurs cannot be taxed and young savers can be taxed only if the lower bound of their endowment is strictly positive. Condition (18) ensures that the demand of public liabilities exceed the supply. Finally, condition (19) limits the quantity of buyback at date 0.

In sum, an equilibrium is such that the flows between private and public sectors are feasible and that F and M play a subgame perfect Nash equilibrium given savers' demands for public liabilities.

When the monetary arithmetic is pleasant, the public sector can earn resources by issuing unbacked reserves and bonds—"bubbles". There are of course many feasible paths for such bubbles, including a non-bubbly one, and we do not want to arbitrarily select a particular pattern, nor do we want to arbitrarily enable the public sector to pick one. Our formalization of the issuance process, whereby savers submit maximum quantities, and our equilibrium requirement that these quantities and the supply of the public sector are merely sustainable, accordingly ensures that we do not arbitrarily rule out any possible (deterministic) pattern of bonds and reserves issuance.

After date 1, in the absence of any tax revenue, the net aggregate real flow received from savers by the public sector at date t,

$$\phi_t \frac{X_t + B_t}{P_{t+1}} - \frac{X_{t-1} + B_{t-1}}{P_t}, \tag{20}$$

is a bubble starting at date t. Strictly positive bubbles clearly exist if and only if the arithmetic is pleasant ($T \leq \delta/\rho$).

The rest of the section characterizes such equilibria, considering in turn the cases of pleasant and unpleasant monetary arithmetics. In order to ease exposition, we study the limiting case in which both authorities prefer any outcome to sovereign default:

$$\alpha_F = \alpha_M = +\infty. \tag{21}$$

Section 3.5.1 shows that our insights carry over with finite default costs.

3.3 Unpleasant arithmetic

Suppose first that $T > \delta/\rho$. We have seen that all public liabilities must be backed by fiscal resources in this case. Given that there are no tax revenues at other dates than 1, any equilibrium must be such that for all $t \geq 1$,

$$\bar{X}_t = \bar{B}_t = 0, \tag{22}$$

and the equilibria are as follows.

Proposition 1. ***(Game of chicken under unpleasant arithmetic)*** *Every equilibrium is default-free and has the following characteristics. Let (b, x) two positive numbers such that $b + x \leq \tau$ and $(b, x) \neq (\tau, 0)$. There exists a unique equilibrium associated with (b, x). It is such that $\bar{B}_0 = B_0 = bP_1$ and $\bar{X}_0 = X_0 = xP_1$. Reciprocally, to every equilibrium corresponds such a pair (b, x).*

- *At date 0, M sets $P_0 = P_M$ and uses its resources $\phi_0 x$ to buyback all or part of L. F uses its resource $\phi_0 b$ to subsidize young entrepreneurs.*
- *At date 1, M sets $P_1 = \max\{P_M; L/(\tau - b)\}$. The public resources τ serve to pay back b, x and $(L/P_1 - x)^+$. There are no residual resources available for transfers to young entrepreneurs if $P_1 > P_M$.*

- *At all dates $t \geq 2$, the price level is P_M and there are no transfers between the public and the private sector.*

If $(b, x) = (\tau, 0)$, then F must buy back the entire liability L at date 0, and P_1 can reach any level above $\max\{P_M; L/\tau\}$.

Proof. See Appendix A.1. □

The fiscal authority seeks to induce the largest possible date-1 price level so as to reduce the real value of L and thus devote the maximum amount of fiscal resources to subsidizing young entrepreneurs. The monetary authority by contrast prioritizes price stability. Accordingly, both supply as much liquidity as possible at date 0. The latter authority uses the proceeds to buyback L whereas the former spends them. The case in which $(b, x) = (\tau, 0)$ is degenerate. When F can borrow against its entire future fiscal resources, it must buyback L entirely to avoid default. The real cost of this buyback depends on savers' (self-justified) anticipation of the date-1 price level.

Strategic fiscal irresponsibility. In sum, F can make a strategic use of fiscal irresponsibility that forces M to accommodate at lower debt levels L than τP_M, the debt capacity of the public sector at the target price level P_M. By spending its entire initial resources $\phi_0 b$ on young entrepreneurs rather than on reimbursing L, F ensures that the date-1 outstanding debt is sufficiently large relative to the date-1 resources of the public sector that M has no other option but accommodating with $P_1 > P_M$ as soon as $L > [\tau - b]P_M \leq \tau P_M$. Such strategic fiscal irresponsibility is all the more effective because b is a large fraction of τ.

The authority that preempts liquidity imposes its views. As a result, across all equilibria (b, x), the utility of M is (weakly) decreasing in b whereas that of F is (weakly) increasing in it. Similarly, for a fixed total public liquidity supply at date 0 $\phi_0(b + x) \leq \phi_0 \tau$, each authority prefers the equilibrium that grants her the largest share of that total date-0 liquidity. Another way of saying this is that if the issuance mechanism was such that one authority could move first and make a take-it-or-leave it offer to date-0 savers, then it would absorb the entire (unit) date-0 savings. In sum, the authority that preempts liquidity can impose its views.

3.4 Pleasant arithmetic

We now characterize equilibria when the monetary arithmetic is pleasant ($T \leq \delta/\rho$). We show that both insights above—i) F is strategically fiscally irresponsible, and ii) the authority that preempts liquidity imposes its views—hold in exactly the same way as when the arithmetic is unpleasant. The only difference with the unpleasant case is that the public sector now has the ability to collect additional resources from the issuance of unbacked securities. How F and M strategically use these resources remains however unchanged.

For ease of exposition only, we focus on no fiscal resources by assuming $\tau = 0$. In addition, we consider only equilibria such that there is a demand for reserves at each date: For all $t \geq 0$, $\bar{X}_t > 0$. We deem such equilibria "liquid". Section 3.5.2 shows that our insights still hold over all possible equilibria.

Proposition 2. ***(Characterization of liquid equilibria)*** *Let $\Lambda = (b_t, x_t)_{t \geq -1}$ a sequence of positive numbers such that $b_{-1} = x_{-1} = 0$ and for all $t \geq 0$,*

$$x_t > 0, \tag{23}$$

$$b_{t-1} + x_{t-1} \leq \phi_t(b_t + x_t) \leq 1. \tag{24}$$

There exists a unique liquid equilibrium associated with Λ. It is such that $\bar{B}_t = B_t = b_t P_{t+1}$ and $\bar{X}_t = X_t = x_t P_{t+1}$. Reciprocally, to every liquid equilibrium corresponds a sequence Λ that satisfies (24).

Every liquid equilibrium is default-free and has the following characteristics:

- *At date 0, M sets $P_0 = P_M$ and uses its resources $\phi_0 x_0$ to buyback L whereas F uses its resources $\phi_0 b_0$ to subsidize young entrepreneurs.*
- *At date 1, M sets $P_1 = \max\{P_M; L/[\phi_1(b_1 + x_1) - b_0]\}$. The public resources $\phi_1(b_1 + x_1)$ serve to pay back b_0, x_0, and $(L/P_1 - x_0)^+$.*
- *At all dates $t \geq 2$, the price level is P_M and the net resources (20) are transferred to current young entrepreneurs.*

Proof. See Appendix A.2. □

Note that the features of the equilibria at dates 0 and 1 are verbatim that in the case of unpleasant monetary arithmetic, up to the only difference that the fiscal resource τ is replaced by the equilibrium surplus from supplying liquidity at date 1, $\phi_1(b_1 + x_1)$. The public sector collects resources from issuing "bubbles" at possibly any date whereas it can only issue date-0 securities backed by date-1 taxes when the arithmetic is unpleasant. The multiplicity of possible bubbly paths drives that of liquid equilibria:

Multiplicity is only due to the multiplicity of bubble patterns. The public sector obtains resources by issuing (deterministic) bubbles, and there are clearly many ways in which such bubbles can arise and shrink over time given a pleasant arithmetic. Proposition 2 states that there is a one-to-one mapping between the bubble patterns such that $x_t > 0$ and the set of liquid equilibria. Thus one can index the set of liquid equilibria with their associated bubble patterns Λ.

Strategic fiscal irresponsibility. For a given Λ, if $L > \bar{L} = \phi_1(b_1 + x_1)P_M$, then the public sector has no choice but inflating away L at date 0 and setting $P_1 > P_M$. It is still the case that F can make a strategic use of fiscal irresponsibility that forces M to accommodate at lower debt levels than $\bar{L}$. By spending its entire initial resources $\phi_0 b_0$ on young entrepreneurs rather than on reimbursing L, F ensures that M has no other option but accommodating with $P_1 > P_M$ as soon as $L > [\phi_1(b_1+x_1) - b_0]P_M \leq \bar{L}$. Such strategic fiscal irresponsibility is all the more effective because b_0 is a large fraction of $\phi_1(b_1 + x_1)$.

The authority that preempts liquidity imposes its views. This can be formally stated as follows when the monetary arithmetic is pleasant. Let $\mathcal{E}((b_t + x_t)_{t\geq 1})$ denote the set of liquid equilibria that share the same value of aggregate public liquidity supply $b_t + x_t$ from date 1 on.

Proposition 3. ***(Fiscal and monetary preferences over liquidity distribution)*** *Over $\mathcal{E}((b_t + x_t)_{t\geq 1})$, the payoff of M is (weakly) decreasing in b_0 whereas that of F is (weakly) increasing in it.*

Proof. In any equilibrium, M sets the price at P_M at all dates but 1 at which $P_1 = \max\{P_M; L/[\phi_1(b_1 + x_1) - b_0]\}$. F can spend $\phi_0 b_0$ at date 0 and, viewed from date 0, $\phi_0 \mid \phi_1(b_1+x_1) - b_0 - L/P_1 \mid^+$ at date 1. This implies the claimed variations with respect to b_0 for their respective preferences. $\square$

Proposition 3 formalizes the idea that the authority that preempts the resources from supplying public liquidity can impose its policy views—more spending for F, a stable price level for M.

A straightforward corollary is that if two equilibria are such that the aggregate inflow streams $(b_t+x_t)_{t\geq 0}$ are ranked, then M may well be better off in the one with the smallest aggregate inflows if it attracts a larger fraction of them at date 0.

These results lead overall to conclude that even though it may expand the set of feasible fiscal and monetary policies, *public liquidity demand does not strengthen the independence of the central bank.* This would be the case only if M was the dominant supplier of liquidity in the following sense: If, for some reason that is beyond the scope of our model, savers were coordinating only on equilibria such that b_0 is sufficiently small that fiscal irresponsibility does not pay off. *The current situation in many countries seems by contrast better described by equilibria in which both F and M are able to extract significant convenience yields from their securities.*

3.5 Extensions

Let us now extend our results to situations where the government can default – i.e. the central bank and the government have only finite default costs – and to situations where the equilibrium can be illiquid – i.e. situations where the demand for liquid assets can be 0 in some periods.

3.5.1 Sovereign default

Whereas the assumption of infinite costs of default simplified the exposition, the important insights carry over when these costs are finite. Suppose $\alpha_M, \alpha_F > 0$. For brevity and realism, we restrict the analysis to the case in which $\alpha_F > 1$. As will be clear below, this implies that M is always the authority that pulls the trigger on default by refusing arbitrarily large levels of inflation. The liquid equilibria have the very same structure as in the limiting case of infinite default costs, the only difference being that strategic fiscal irresponsibility is less effective at leading M to chicken out.

Proposition 4. ***(Liquid equilibria with finite default costs)*** *There is a one-to-one mapping between the set of liquid equilibria and that of the sequences Λ that satisfy (23),(24), and $L \leq \phi_1(b_1 + x_1)(P_M + \alpha_M)$. The equilibria are as described in Proposition 2 except when $[\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M) < L \leq \phi_1(b_1 + x_1)(P_M + \alpha_M)$. In this case the equilibrium is such that M accommodates as much as possible: $P_1 = P_M + \alpha_M$, and F does not spend all of $\phi_0 b_0$ at date 0, but uses part of it to extinguish L instead.*

Proof. See Appendix A.2. □

Default as an alternative to hyperinflation. The fiscal authority's cost from sovereign default α_F plays no role in equilibrium determination. This owes to the assumption that it is larger than the maximum gross resources that F could generate by defaulting at any date ($\alpha_F > 1$). This implies that default in equilibrium is always triggered by M, which does so when all the current real resources of the public sector do not suffice to repay the outstanding liabilities without a rate of inflation larger than α_M/P_M. In other words, default in this economy is a strategic decision of the central bank, who prefers this option to the always available one of massive money printing.

The smaller α_M, the less M chickens out. Unsurprisingly, the strategy of fiscal irresponsibility of F is less effective, the less M cares about default. The new equilibrium feature introduced by finite default costs is that M is not willing to generate whichever date-0 inflation averts sovereign default. Its indifference point between inflation and outright default is at the price level $P_1 = P_M + \alpha_M$. Anticipating this, F cannot spend its entire initial surplus $\phi_0 b_0$ when L is sufficiently large. It must instead reimburse some of L at date 0 so as to ensure that M is exactly at its indifference point at date 1. In particular, Proposition 4 confirms that strategic fiscal irresponsibility is totally ineffective if $\alpha_M = 0$.

3.5.2 Illiquid equilibria

Our restriction to liquid equilibria simplifies the exposition but is admittedly arbitrary. Here we sketch how lifting it affects the analysis. Appendix A.2 develops the full-fledged analysis. Unlike under the restriction to liquid equilibria, there may now be several equilibria associated with a given sequence $\Lambda = (b_t, x_t)_{t \geq -1}$ as soon as the real demand for reserves x_t is equal to 0 for some dates $t \geq 0$.[6] Let $\Lambda = (b_t, x_t)_{t \geq -1}$ a sequence of positive numbers such that $b_{-1} = x_{-1} = 0$ and for all $t \geq 0$,

$$b_{t-1} + x_{t-1} \leq \phi_t(b_t + x_t) \leq 1. \tag{25}$$

We also denote λ_t the net aggregate real flow (20) received from savers at date t. The equilibria associated with Λ have the following features:

- If $x_0 = b_0 = x_1 = b_1 = 0$ then the public sector defaults at date 1 (but not afterwards). Otherwise equilibria are default-free and:
- If $\lambda_t = 0$ for some $t > 1$, then for some histories of the game up to t, any price $P_t \geq P_M$ is an equilibrium outcome. There are also histories, detailed in Appendix A.2, such that M can impose $P_t = P_M$.
- If $x_0 = \lambda_1 = 0$, then any price above $\max\{P_M; L/b_0\}$ is an equilibrium outcome;
- Otherwise all equilibria associated with Λ are identical at date t, at which date they have the same features as liquid equilibria.

Two new features arise when the demand for reserves may dry up at any date. First, if the public sector is completely illiquid before date 1, then it has no choice but defaulting at date 0. Second and more interestingly, the possibility that M does not receive fresh

[6] This is the exact equivalent under pleasant arithmetic of the situation in which $(b, x) = (\tau, 0)$ under an unpleasant one.

liquidity at every given date creates room for equilibria with higher price levels than P_M. The important bottom line though is that even when Λ is associated with multiple equilibria, the comparative statics with respect to b_0 in Proposition 3 still apply.

4 General conditions for pleasant arithmetic

In this section, we provide general conditions under which the monetary arithmetic is pleasant. To this purpose, we first introduce a general setting. In this setting, we define and characterize pleasant monetary arithmetic and we show how this affects the set of feasible price levels that the monetary authority can target. We finally extend our results to multiple public liabilities – government debt and central bank reserves.

4.1 General setup

A policy (P, s) is a pair of sequences of real numbers $P = (P_t)_{t\in\mathbb{N}}$ and $s = (s_t)_{t\in\mathbb{N}}$. The sequence P represents the public sector's choice of price levels, and s stands for the sequence of real fiscal surpluses. We seek to identify the set of feasible policies of an economy, that is, the policies (P, s) that are compatible with market clearing and optimization by the private sector. We formally proceed as follows. Let $D_{-1} > 0$. For every $t \in \mathbb{N}$, let ϕ_t be a mapping from $\mathbb{R}^+$ into $\mathbb{R}^+ \setminus \{0\}$, $\bar{\phi}_t \leq \phi_t$ a mapping from $\mathbb{R}^+$ into itself, and S_t a non-empty subset of $\mathbb{R}$.

Definition 2. ***(Feasible policy)*** *Given* $\left(D_{-1}, \left(S_t, \phi_t, \bar{\phi}_t\right)_{t\in\mathbb{N}}\right)$, *a policy* (P, s) *is feasible if*

(i) For all $t \in \mathbb{N}$, $P_t > 0$ *and* $s_t \in S_t$.

(ii) There exists a sequence of real numbers $(D_t)_{t\in\mathbb{N}}$ *that satisfies for all* $t \in \mathbb{N}$:

$$\frac{D_{t-1}}{P_t} = s_t + \phi_t\left(\frac{D_t}{P_{t+1}}\right)\frac{D_t}{P_{t+1}}, \tag{26}$$

$$\lim_{\tau\to\infty}\left(\prod_{i=t}^{\tau}\bar{\phi}_i\left(\frac{D_i}{P_{i+1}}\right)\right)\frac{D_\tau}{P_{\tau+1}} = 0. \tag{27}$$

We denote by $\mathcal{F}$ *the set of such feasible policies.*

The above abstract definition subsumes feasible fiscal and monetary policies in many models. The parameter D_{-1} represents nominal public liabilities inherited from an unmodelled past and due at date 0 – as L_{-1} in Sections 2 and 3. For every $t > 0$, D_t corresponds to a nominal liability—a number of currency units due at date $t + 1$ and issued at date t by the public sector. Conditions (26) and (27) are reduced forms for the restrictions that optimization by the private sector and market clearing impose on feasible fiscal policies s and monetary policies P in a large class of standard economic models. Condition (26) is the date-t budget constraint of the public sector given the pricing ϕ_t of public debt, and (27) is a transversality condition.

In other words, our approach consists in summarizing an economy with the two ingredients that matter for the determination of feasible fiscal and monetary policies, the pricing of public debt $(\phi_t)_{t\in\mathbb{N}}$ and the intertemporal rates of substitution in the transversality condition $(\bar{\phi}_t)_{t\in\mathbb{N}}$. If they are equal then public debt does not offer liquidity services.

OLG example. In this case, $\phi_t(b) = \min\{e^{-\rho}; 1/b\}$ when t is not a multiple of $T+1$ and $\phi_t(b) = \min\{e^{\delta}; 1/b\}$ otherwise. The function $\bar{\phi}_t$ can be normalized to 0 so that (27) always holds given that optimization by short-lived agents requires no transversality condition. The set S_t is $(-\infty, \lambda]$ for $t = 0$ and $(-\infty, 0]$ afterwards.

Debt in the utility function Consider an economy populated by a public sector and a private sector comprised of a representative agent. The agent receives $Y > 0$ consumption units at each date $t \in \mathbb{N}$. She derives utility out of consumption and real holdings of public liabilities. At each date $t \in \mathbb{N}$, the public sector sets the price level P_t, raises a real lump-sum tax s_t, and issues a nominal claim of B_t currency units due at $t+1$. It starts out with an exogenous legacy nominal liability $D_{-1} > 0$ due at date 0.

Denoting q_t the date-t real price of public bonds, the agent selects a consumption stream $(C_t)_{t\in\mathbb{N}}$ and bond holdings $(H_t)_{t\in\mathbb{N}}$ that solve for some $\beta \in (0,1)$

$$\begin{aligned}&\max_{(C_t,H_t)_{t\in\mathbb{N}}} \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v\left(\frac{H_t}{P_{t+1}}\right)\right),\\ &\text{s.t. } \forall t \in \mathbb{N}, Y_t - s_t + \frac{H_{t-1}}{P_t} = q_t \frac{H_t}{P_{t+1}} + C_t.\end{aligned} \tag{28}$$

In addition, goods and bonds market clearing implies that for all t,

$$C_t = Y \text{ and } H_t = D_t. \tag{29}$$

Standard restrictions on u and v (e.g., Kamihigashi, 2003) imply that $(C_t, H_t)_{t\in\mathbb{N}}$ solves (28) and (29) if and only if it satisfies two conditions, a local one (Euler equation):

$$q_t = \beta + \frac{v'\left(\frac{D_t}{P_{t+1}}\right)}{u'(Y)}, \tag{30}$$

and a terminal one (transversality condition):

$$\lim_{k\to\infty} \beta^k \frac{D_{t+k}}{P_{t+k+1}} = 0. \tag{31}$$

In this case, $\phi_t(b) = q_t = \beta + \frac{v'\left(\frac{D_t}{P_{t+1}}\right)}{u'(Y)}$ and $\bar{\phi}_t = \beta$. The set S_t is $\mathbb{R}$ for any $t \geq 0$.

We impose in the remainder of the paper the following restrictions.

Assumption 1. *For every $t \in \mathbb{N}$, the function ϕ_t is continuously decreasing, the function $b \mapsto \bar{\phi}_t(b)/\phi_t(b)$ is increasing and the function $b \mapsto b\phi_t(b)$ is strictly increasing.*

The first item of Assumption 1 imposes that, at any date, the price $\phi_t(b)$ (weakly) decreases with respect to the real level of the newly issued debt b. The second item imposes that this decrease dominates the potential increase of $\bar{\phi}(.)$.[7] Finally, the third item imposes that even though the price of debt decreases with the quantity of newly issued debt, the real amount of fiscal resources increases with it. Note that the OLG and the debt-in-the-utility-function examples satisfy these restrictions.

4.2 Pleasant monetary arithmetic: Definition and characterization

We now investigate the interdependence between fiscal and monetary policy in the general setting that we have introduced. To this purpose, we first formalize the notion of a tradeoff between these two policies — i.e. modifying one side of the policy, either the fiscal or the monetary one, necessarily leads to also modifying the other. We then connect the presence of tradeoffs to the price of public debt and to whether a rollover of debt is possible.

Fiscal-monetary tradeoff A feasible policy features a fiscal-monetary tradeoff—simply a "tradeoff" henceforth— if and only if there exists no other feasible policy with both smaller surpluses and initial price level. Formally,

Definition 3. ***(Tradeoff)*** *A feasible policy (P, s) features a tradeoff if and only if for any policy $(P', s') \in \mathcal{F}$,*

$$P'_0 \leq P_0 \textit{ and } s' \leq s \to P'_0 = P_0 \textit{ and } s' = s. \tag{32}$$

[7]As we will see soon, this means that the convenience yield decreases with the level of newly issued debt.

Starting from a policy that features a tradeoff, a monetary authority averse to sovereign default would have to accommodate with inflation if the fiscal one were willing to reduce taxes, and vice versa. This is the type of situations deemed "game of chicken" by Wallace. The following proposition establishes an equivalence between the presence of a fiscal-monetary tradeoff and the extent to which fiscal policy uniquely pins down the price level.

Proposition 5. ***(A tradeoff implies fiscal determination of the price level)*** *Let* $(P, s) \in \mathcal{F}$ *and* $\mathcal{P}_s = \{P'_0 \mid (P', s) \in \mathcal{F}\}$.

(i) $\mathcal{P}_s$ *is an interval;*

(ii) $\mathcal{P}_s$ *is a singleton if and only if every policy* $(P', s) \in \mathcal{F}$ *features a tradeoff.*

Proof. See Appendix A.3. □

Proposition 5 first shows that the set of feasible initial price levels associated with a given path of surpluses is an interval. This owes to the monotonicity posited by Assumption 1. It then establishes the equivalence between the existence of a tradeoff for every feasible policy (P', s) and the fact that s fully determines the initial price level.

Pleasant monetary arithmetic. Based on our definition of a tradeoff, we now offer a more global concept of fiscal and monetary interdependence.

Definition 4. ***(Pleasant monetary arithmetic)*** *Given* $(S_t, \phi_t, \bar{\phi}_t)_{t\in\mathbb{N}}$*, the monetary arithmetic is unpleasant if for all* $D_{-1} > 0$*, every feasible policy features a tradeoff. Otherwise, the monetary arithmetic is pleasant.*

In other words, the monetary arithmetic is unpleasant when any feasible path for public finances features a tradeoff. The arithmetic is conversely pleasant as soon as there exists an initial value of legacy liabilities $D_{-1} > 0$ for which fiscal and monetary policy are not interdependent this way.

Our characterization of situations of pleasant monetary arithmetic requires the introduction of debt rollovers. Before defining a rollover, let us define a convenience yield.

Definition 5. ***(Convenience yield)*** *We deem* $\delta_t(b)$ *the date-*t *convenience yield associated with real debt* $b > 0$ *defined as*

$$\delta_t(b) = \begin{cases} \log(\phi_t(b)) - \log(\bar{\phi}_t(b)) & \text{if } \bar{\phi}_t(b) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{33}$$

We can now define a rollover of debt.

Definition 6. ***(Rollover)*** *A rollover is a sequence* $(b_t)_{t\in\mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$ *such that for all* $t \in \mathbb{N}$*,*

$$b_t = b_{t+1}\phi_t(b_{t+1}), \tag{34}$$

and such that $\sum_{t\in\mathbb{N}} \delta_t(b_t)$ *diverges.*

The existence of such a rollover is a sufficient and necessary condition for the monetary arithmetic to be pleasant.

Proposition 6. ***(Characterization of pleasant monetary arithmetic)*** *The monetary arithmetic is pleasant if and only if there exists a rollover.*

Proof. See Appendix A.4. □

Put simply, if public liabilities can be rolled over at a cost that is sufficiently low relative to the long-term opportunity cost of capital of the private sector, then the public sector can extract the resulting surplus in order to gain degrees of freedom and relax the interdependence between fiscal and monetary policies.

We actually show in the proof of Proposition 6 that if there exists such a rollover, then the set of initial prices associated with zero primary surpluses ($s = 0$), $\mathcal{P}_0$, is not a singleton regardless of the initial level of debt D_{-1}.

Proposition 7. ***(Fiscal determination at the minimum price level)*** *Suppose that there exists* $\tilde{b} > 0$ *such that there exists no rollover with* $b_0 \geq \tilde{b}$*. Then, given* D_{-1}*,*

(i) If $s' < s$*, then* $\inf \mathcal{P}_{s'} > \inf \mathcal{P}_s$*.*

(i) For any $p > 0$*, there exists a level of surpluses* s *such that* $\inf \mathcal{P}_s = p$*.*

Proof. See Appendix A.5. □

Many observers hold the view that fiscal considerations matter for monetary policy only in times of stretched public finances, or, that the fiscal theory of the price level is practically relevant only during fiscal crises. Proposition 7 translates this in our formal setting.

The first part of the proposition states that even when surpluses do not dictate the price level, there is still a point at which the tradeoff between a lower price level and lower surpluses kicks in again.

Proposition 7 also shows that for any level p, there always exists a level of surpluses leading prices to be above p. In this sense the independence of the central bank is unclear as long as the government can choose the stream of surpluses.

It is instructive to connect the results in Propositions 6 and 7 to the OLG and the debt-in-the-utility examples.

OLG example. In this case, $\bar{\phi}_t = 0$ for all t and so the monetary arithmetic is pleasant if and only if there exists a rollover. As established in Section 2, this is the case if and only if $T \leq \delta/\rho$.

Regarding Proposition 7, it is indeed the case that rollovers are not possible starting from a real value above $\tilde{b} = 1$. As stated in the Proposition, the smallest feasible price level $B_{-1}/(s_0 + 1)$ is strictly decreasing in s.

DIU example. According to the formalism of subsection 4.1, $\bar{\phi}_t$ is a constant equal to β, and ϕ_t is given by the right-hand side of (30). For all $t \in \mathbb{N}$, $S_t = \mathbb{R}$.

We will study here the case in which

$$v(h) = \alpha u'(Y) \frac{h^{1-\gamma}}{1-\gamma}, \tag{35}$$

where $\alpha > 0$ and $\gamma \in (0,1)$. The existence of a rollover is not an issue in this case since

$$b\phi_t(b) \equiv \Psi(b) = \beta b + \alpha b^{1-\gamma} \tag{36}$$

is a bijection over $\mathbb{R}_+$. In fact, it is possible to construct a rollover starting from any initial value. The question is then whether there exists one with a diverging sum of convenience yields. We have

$$\forall t \in \mathbb{N}, \delta_t(b) = \log\left(1 + \frac{\alpha b^{-\gamma}}{\beta}\right). \tag{37}$$

Let

$$b^* = \left(\frac{\alpha}{1-\beta}\right)^{\frac{1}{\gamma}} \tag{38}$$

the unique non-zero fixed point of Ψ. We show in Appendix YYY that a rollover has a diverging sum of convenience yields if and only if $b_0 \in (0, b^*]$. For such values of b_0 the rollover converges to a finite value, whereas for $b_0 > b^*$ it tends to $+\infty$ so quickly that the convenience yields decrease sufficiently fast for $\sum \delta_t$ to converge .

This implies first that there exists a rollover with a diverging sum of convenience yields, and so that the monetary arithmetic is pleasant. This also implies that for a given surplus stream of the form $s = \left(\mathbb{1}_{\{t=0\}}x\right)_{t\in\mathbb{N}}$ for some $x > 0$, we have

$$\inf \mathcal{P}_s = \frac{D_{-1}}{x + b^*}, \tag{39}$$

which decreases in x and in the preference for liquidity α, and increases in the elasticity of the convenience yield γ: the interdependence between fiscal and monetary policy is thus tighter when the demand for public liquidity is more elastic.

Note that in the extreme case where $\gamma = 1$, that is $v(h) = \alpha \log h$, where $\alpha > 0$, then, (28), (30) and (31) lead to

$$\frac{B{-1}}{P_0} = \sum_{t\in\mathbb{N}} \beta^t s_t + \frac{\alpha}{u'(Y)(1-\beta)}. \tag{40}$$

This means that there is a strong interdependence between monetary and fiscal policies: as in the fiscal theory of the price level, the present value of future surpluses uniquely pins down the price level, but at a lower level compared with the fiscal theory because of the convenience yield. One can observe that a large value to hold public liabilities (as measured by the parameter α) leads to a lower price level P_0 through equation (40).

4.3 Extension to multiple public liabilities

This subsection extends the analysis of feasible policies to the case where the public sector may issue two types of liabilities, government debt and central-bank reserves and where the central bank and the government have separate budget constraint as in Bassetto and Messer (2013) or Hall and Reis (2015).

Feasible policies with central-bank reserves We now assume that the public sector includes a monetary authority that can issue reserves. We denote by $X_t \geq 0$ the amount of reserves issued at date t and due at date $t+1$ and by ϕ_t^X the real price of reserves.[8] We suppose that ϕ_t^X and the price of bonds ϕ_t may a priori depend on the real holdings of both bonds and reserves by the private sector.

The monetary authority may use the proceeds from issuing reserves to trade government bonds or/and transfer all or part of them to the government. Let D_t^B denote the date-t debt holding of the central bank, and d_t denote the date-t real remittances to the

[8] For expositional simplicity and symmetry, we model central-bank reserves as one-period claims akin to government bonds. Introducing reserves with indefinite maturity as they are in practice would not add any insight.

government. These transfers can be positive (dividends) or negative (recapitalization).

Definition 7. ***(Feasible extended policy)*** *Given $\left(D_{-1}, D_{-1}^B, X_{-1}, (S_t, \phi_t, \phi_t^X, \bar{\phi}_t)_{t\in\mathbb{N}}\right)$, an extended policy (P, s, d) is feasible if*

(i) For all $t \in \mathbb{N}$, $P_t > 0$ and $s_t \in S_t$.

(ii) There exists a triplet of sequences of positive real numbers $(D_t, X_t, D_t^B)_{t\in\mathbb{N}}$ that satisfies for all $t \in \mathbb{N}$:

$$\frac{D_{t-1}}{P_t} = s_t + d_t + \phi_t((D_t - D_t^B)/P_{t+1}, X_t/P_{t+1})\frac{D_t}{P_{t+1}}, \tag{41}$$

$$\begin{aligned}&\phi_t((D_t - D_t^B)/P_{t+1}, X_t/P_{t+1})\frac{D_t^B}{P_{t+1}} + \frac{X_{t-1}}{P_t} + d_t = \\ &\frac{D_{t-1}^B}{P_t} + \phi_t^X((D_t - D_t^B)/P_{t+1}, X_t/P_{t+1})\frac{X_t}{P_{t+1}},\end{aligned} \tag{42}$$

$$\lim_{\tau\to\infty}\left(\prod_{i=t}^{\tau}\bar{\phi}_i((D_i - D_i^B + X_i)/P_{i+1})\right)\frac{D_\tau - D_\tau^B + X_\tau}{P_{\tau+1}} = 0, \tag{43}$$

$$D_t^B \leq D_t. \tag{44}$$

We denote by $\mathcal{F}^X$ the set of such feasible extended policies.

Equation (41) is simply the budget constraint of the government (26) where remittances from the central bank are added to surpluses. Equation (42) is the budget constraint of the central bank. Equation (43) is the terminal condition. Finally, equation (44) imposes that the bonds' holding of the central bank be lower than the total stock of government bonds.

For a given stream of surpluses s, we denote by $\mathcal{P}_s^X$ the set of initial price levels that belong to a feasible extended policy: $\mathcal{P}_s^X = \{P_0 | (P, s, d) \in \mathcal{F}^X\}$. We now study how the availability of such extended policies expands the set of circumstances in which monetary arithmetic is pleasant. We study in turn the polar cases in which by contrast only reserves carry a convenience yield, and in which bonds and reserves are perfect substitutes.

When bonds and reserves are imperfect substitutes Suppose that reserves are the only asset providing liquidity services leading to a disconnect between the price of reserves and that of bonds. For simplicity, we assume that $\phi_t = \bar{\phi}_t$ does not depend on the quantity of public liabilities and that ϕ_t^X only depends on real holdings of reserves. More precisely, at any date t:

$$\phi_t = \bar{\phi}_t \text{ and } \phi_t^X(X_t/P_{t+1}) \geq \phi_t \text{ with } \phi_t^X(0) > \phi_t. \tag{45}$$

In this context, it is useful to extend our definitions of rollover and convenience yield to central bank's reserves. A *rollover for reserves* is a sequence $(x_t)_{t\in\mathbb{N}} \in (0,+\infty)^{\mathbb{N}}$ such that for all $t \in \mathbb{N}$,

$$x_t = x_{t+1}\phi_t^X(x_{t+1}). \tag{46}$$

We deem $\delta_t^X(x)$ the date-t convenience yield associated with real reserves $x > 0$ defined as

$$\delta_t^X(x) = \begin{cases} \log(\phi_t^X(x)) - \log(\bar{\phi}_t) & \text{if } \bar{\phi}_t > 0, \\ 1 & \textit{otherwise}. \end{cases} \tag{47}$$

Importantly, a rollover for reserves is expressed in real terms. In nominal terms, reserves can always be rolled over, as reserves are always reimbursed using reserves.

Proposition 8. ***(Central bank's issuance of reserves may relax the fiscal-monetary tradeoff)*** *Suppose $X_{-1} = D_{-1}^B$. If there exists a rollover of reserves $\{x_t\}_{t\geq 0}$ such that $\sum_{t\in\mathbb{N}} \delta_t^X(x_t)$ diverges, then for any feasible policy (P, s), there exists a feasible extended policy (P', s', d) with $P_0 < P_0'$ and $s \leq s'$.*

Proof. See Appendix A.6. □

When reserves can be rolled-over then the issuance of reserves generates additional real revenue for the central bank. The central bank can issue such liabilities either through monetary financing of the deficit or through open market operations. In any case, this additional revenue will be ultimately rebated to the government leading to either lower price level or lower surpluses. The existence of a convenience yield on reserves thus potentially relaxes the interdependence between the two authorities.

However, the conclusions of Section 3 still remain: the additional resources obtained by the public sector through the issuance of reserves may not necessarily ensure the independence of the central bank. The government can force the central bank to transfer these additional resources and then constraint the central bank to give up on the price stability objective.

Remark. Proposition 8 contrasts with the unpleasant fiscal arithmetic described by Sargent and Wallace (1981). In their paper, an additional deficit necessarily leads to higher price level. Indeed, when the government increases deficit, the only feasible monetary policy is to raise the stock of money eventually collecting higher seigneurage revenues to balance the government budget. In their paper, a quantity theory demand for money with constant income velocity links the quantity of money and the price level and therefore larger deficit causes higher prices. Because, we do not assume such a link here, the central bank can increase reserves, generating additional revenues, without triggering higher price level.

When bonds and reserves are perfect substitutes Let us now turn to the case where bonds and reserves are perfect substitutes. This happens when they share the same price that depends on the total public liability held by the private sector, $\tilde{D}_t = D_t - D_t^B + X_t$, that is:

$$\text{For all } t \in \mathbb{N},\ \phi_t^X(\tilde{D}_t) = \phi_t(\tilde{D}_t). \tag{48}$$

Note that this corresponds to a situation where the ZLB binds or, more generally, a situation where the central bank is already supplying a large amount of reserves.

The budget constraint of the central bank and that of the government can then be consolidated without loss of generality:

$$\frac{\tilde{D}_{t-1}}{P_t} = s_t + \phi_t(\tilde{D}_t/P_{t+1})\frac{\tilde{D}_t}{P_{t+1}}. \tag{49}$$

It follows that only the path of aggregate public liabilities held by the private sector matters for price-level determination.

Proposition 9. ***(Central bank's balance sheet irrelevance for price determination)*** *If reserves and bonds are perfect substitutes, then for a given level of public liability held by the private sector $\tilde{D}_{-1}$ and fiscal surpluses s, the set of initial feasible prices is unaffected by the central bank's balance-sheet structure: $\mathcal{P}_s = \mathcal{P}_s^X$.*

Proof. See Appendix A.7. □

Proposition 9 shows that, given surpluses, the set of initial feasible prices is unaffected by the balance-sheet tools of the central bank. In particular, the path of remittances d is irrelevant for the determination of the price level because the government can adjust the issuance of debt such as to exactly offset the newly issued reserves, letting the quantity of debt held by the public unchanged. Therefore, when reserves and bonds are perfect substitute, monetary financing of the deficit has no impact on the price level. Note that this is true regardless of whether the monetary arithmetic is pleasant or not.

In addition, open-market operations at any date (defined here as the case $X_t = D_t^B$) do not change the public liability $\tilde{D}$ and do not affect the set of feasible policies (P, s). This result echoes the well-known irrelevance result of open market operations (Wallace, 1981; Chamley and Polemarchakis, 1984). Indeed, if reserves and government bonds do not provide different liquidity services then open market operations do not alter the aggregate budget constraint of the public sector vis-à-vis the private sector.

An implication of Proposition 9 is that when a rollover is impossible (see Proposition 6), then the only initial price level that is feasible is the singleton $\mathcal{P}_s$ that only depends on the legacy debt $\tilde{D}_{-1}$ and the stream of surpluses s but is completely independent of the amount of reserves and of central bank's debts holding. In a similar environment, Benigno (2017) establishes the related result that if the government passively passes through the remittances to the public (in our context, if $s = -d$), the central bank can pin down the price level because it de facto decides on the stream of fiscal surpluses.

References

Asriyan, V., L. Fornaro, A. Martín, and J. Ventura (2019): "Monetary Policy for a Bubbly World," CEPR Discussion Papers 13803, C.E.P.R. Discussion Papers.

Bassetto, M. and W. Cui (2018): "The fiscal theory of the price level in a world of low interest rates," *Journal of Economic Dynamics and Control*, 89, 5–22.

Bassetto, M. and T. Messer (2013): "Fiscal Consequences of Paying Interest on Reserves," *Fiscal Studies*, 34, 413–436.

Bénassy, J.-P. (2008): "The fiscal theory of the price level puzzle: a non-Ricardian view," *Macroeconomic Dynamics*, 12, 31–44.

Benigno, P. (2017): "A Central Bank Theory of Price Level Determination," CEPR Discussion Papers 11966, C.E.P.R. Discussion Papers.

Bianchi, F., H. Kung, and T. Kind (2019): "Threats to Central Bank Independence: High-Frequency Identification with Twitter," NBER Working Papers 26308, National Bureau of Economic Research, Inc.

Blanchard, O. (2019): "Public Debt and Low Interest Rates," *American Economic Review*, 109, 1197–1229.

Buiter, W. H. (2002): "The Fiscal Theory of the Price Level: A Critique," *The Economic Journal*, 112, 459–480.

Chamley, C. and H. Polemarchakis (1984): "Assets, General Equilibrium and the Neutrality of Money," *Review of Economic Studies*, 51, 129–138.

Cochrane, J. H. (2001): "Long-Term Debt and Optimal Policy in the Fiscal Theory of the Price Level," *Econometrica*, 69, 69–116.

Cukierman, A. (2008): "Central bank independence and monetary policymaking institutions – Past, present and future," *European Journal of Political Economy*, 24, 722–736.

Del Negro, M. and C. A. Sims (2015): "When does a central bank?s balance sheet require fiscal support?" *Journal of Monetary Economics*, 73, 1–19.

Diamond, P. A. (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55, 1126–1150.

Farhi, E. and J. Tirole (2012): "Bubbly Liquidity," *Review of Economic Studies*, 79, 678–706.

Farmer, R. E. and P. Zabczyk (2019): "The Fiscal Theory of the Price Level in Overlapping Generations Models," NBER Working Papers 25445, National Bureau of Economic Research, Inc.

Gali, J. (2014): "Monetary Policy and Rational Asset Price Bubbles," *American Economic Review*, 104, 721–52.

Hall, R. E. and R. Reis (2015): "Maintaining Central-Bank Financial Stability under New-Style Central Banking," CEPR Discussion Papers 10741, C.E.P.R. Discussion Papers.

Holmström, B. and J. Tirole (1998): "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106, 1–40.

Kamihigashi, T. (2003): "Necessity of transversality conditions for stochastic problems," *Journal of Economic Theory*, 109, 140–149.

Krishnamurthy, A. and A. Vissing-Jorgensen (2012): "The Aggregate Demand for Treasury Debt," *Journal of Political Economy*, 120, 233–267.

Leeper, E. M. (1991): "Equilibria under 'active' and 'passive' monetary and fiscal policies," *Journal of Monetary Economics*, 27, 129–147.

Martin, A. and J. Ventura (2012): "Economic Growth with Bubbles," *American Economic Review*, 102, 3033–3058.

McCallum, B. T. (2001): "Indeterminacy, bubbles, and the fiscal theory of price level determination," *Journal of Monetary Economics*, 47, 19–30.

Niepelt, D. (2004): "The Fiscal Myth of the Price Level," *The Quarterly Journal of Economics*, 119, 277–300.

Sargent, T. J. and N. Wallace (1981): “Some unpleasant monetarist arithmetic,” *Quarterly Review.*

Sims, C. A. (1994): “A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy,” *Economic Theory*, 4, 381–399.

——— (2003): “Fiscal Aspects of Central Bank Independence,” Princeton University.

Tirole, J. (1985): “Asset Bubbles and Overlapping Generations,” *Econometrica*, 53, 1499–1528.

Wallace, N. (1981): “A Modigliani-Miller Theorem for Open-Market Operations,” *American Economic Review*, 71, 267–274.

Woodford, M. (1990): “Public Debt as Private Liquidity,” *American Economic Review*, 80, 382–388.

——— (1994): “Monetary policy and price level determinacy in a cash-in-advance economy,” *Economic theory*, 4, 345–380.

——— (1995): “Price-level determinacy without control of a monetary aggregate,” *Carnegie-Rochester Conference Series on Public Policy*, 43, 1–46.

——— (2001): “Fiscal Requirements for Price Stability,” *Journal of Money, Credit and Banking*, 33, 669–728.

Appendix

A Proofs

A.1 Proof of Proposition 1

After date 2. F and M can only issue securities at date 0 that are backed by the date-1 fiscal resources τ. Thus there is no action after date 2 other than M setting the price level at P_M.

No default in equilibrium. Suppose an equilibrium is such that the public sector defaults at date 1. Default should be total given the fixed cost of doing so, and thus liquidity demand such that $\bar{X}_0 = \bar{B}_0 = 0$. But in this case M would set $P_1 = L/\tau$ and avoid default, a contradiction.

Equilibria when $(b, x) \neq (\tau, 0)$**.** We show that there is exactly one equilibrium associated with such (b, x). We denote $\bar{P}_1$ the savers' anticipation of the date-1 price at date 0, X and B the respective (nominal) supply of liquidity by M and F at date 0, and L_M and L_F the respective (nominal) shares of L that they prepay at date 0. An equilibrium is such that

$$\bar{X}_0 = x\bar{P}_1, \tag{50}$$

$$\bar{B}_0 = b\bar{P}_1, \tag{51}$$

$$0 \leq L_M \leq X \leq \bar{X}_0, \tag{52}$$

$$0 \leq L_F \leq B \leq \bar{B}_0, \tag{53}$$

$$0 \leq L_F + L_M \leq L. \tag{54}$$

Moving first at date 1, M sets P_1 at the largest of two values, either P_M or the smallest P_1 that ensures solvency:

$$P_1\tau \geq L - L_F - L_M + X + B. \tag{55}$$

Rationally anticipating this at date 0, F maximizes $B - L_F$ by setting $B = \bar{B}_0$ and $L_F = 0$, whereas M sets $X - L_M = 0$. Thus

$$P_1 = \max\left\{P_M; \frac{L + \bar{B}_0}{\tau}\right\} = \max\left\{P_M; \frac{L}{\tau - b}\right\} \tag{56}$$

Equilibria when $(b, x) = (\tau, 0)$. In this case,

$$\tau P_1 \geq L - L_F + \tau \bar{P}_1 \tag{57}$$

implies that equilibria must be such that $L_F = L$. In this case, any savers' beliefs $P_1 \geq \max\{P_M; L/\tau\}$ can be sustained in equilibrium.

A.2 Proof of Propositions 2, 4, and of the results in Section 3.5.2

We characterize the equilibria in the general case in which $\alpha_M > 0$, $\alpha_F > 1$, and allowing for zero demand for reserves at any date—that is, without restricting the analysis to liquid equilibria. The results in Section 3.5.2 then obtain by letting $\alpha_F, \alpha_M \to +\infty$, that in Proposition 4 by restricting the analysis to equilibria such that $x_t > 0$ for all $t \geq 0$, and that in Proposition 2 by doing both.

Note first that any equilibrium must be such that $P_t \geq P_M$ for all $t \geq 2$. Suppose by contradiction that $P_t < P_M$. Then M can raise the price to P_M thereby both reaching its target and reducing the burden of debt repayment, and savers should anticipate this at $t - 1$.

At dates $t \geq 2$.

- If $\lambda_t > 0$, then the equilibrium must be such that $P_t = P_M$. Suppose by contradiction that $P_t > P_M$. Then M can reduce the price, possibly but not necessarily all the way to P_M, using all or part of λ_t to pay for the induced increase in real debt due.

- If $\lambda_t = 0$, suppose that there exists $k \geq 1$ such that $x_{t-k}, x_{t-k+1}, ..., x_{t-1} > 0$ and $\lambda_{t-k} > 0$. (Note that such a k always exists when the equilibrium is liquid since $x_0 = \lambda_0 > 0$). In this case it must also be that $P_t = P_M$: Otherwise M can optimally leave an arbitrarily small fraction of liquidity demand $\bar{X}_{t-k}$ unsatisfied, and so on at each date until $t - 1$ thereby ensuring $\lambda_t > 0$ in which case it can reduce the price as seen above. If such a k does not exist then any $P_t \in [P_M, P_M + \alpha_M]$ is a sustainable equilibrium outcome, unless of course $x_{t-1} = b_{t-1} = 0$ in which case M can enforce $P_t = P_M$ since there are no outstanding liabilities at date t.

At dates 0 and 1.

- Note first that the public sector defaults on L at date 1 if and only if $\phi_1(x_1 + b_1)(P_M + \alpha_M) < L$, in which case $x_0 = b_0 = 0$. Default comes at fixed costs for both authorities and so default is total when it occurs, which savers anticipate at date 0.

Otherwise, in the absence of default at date 1,

- If $\phi_1(b_1 + x_1) = b_0$ then it must be that $x_0 = 0$ and $b_0 > 0$. Any date-1 price $P_1 \in [\max\{P_M; L/b_0\}, P_M + \alpha_M]$ is a sustainable equilibrium outcome in this case, implying that it must be that $b_0 \geq L/(P_M + \alpha_M)$.

- Suppose $\phi_1(b_1+x_1) > b_0$ and let $L' \leq L$ the nominal value of the exogenous liability that remains to be repaid at date 1. It must be that $L' \leq \lambda_1(P_M+\alpha_M)$, and M sets the date-1 price at the minimum level that averts default, $P_1 = \max\{P_M; L'/(P_M + \alpha_M)\}$. This implies in turn that at date 0, it is weakly dominant for M to minimize L' and thus to use x_0 to prepay as much of L as possible. Conversely, F seeks to induce the highest possible value of P_1 and thus only prepays the minimum amount that averts date-0 default, spending the residual on young date-(0) entrepreneurs. As a result,

  - If $L \leq [\phi_1(b_1+x_1)-b_0]P_M$ then the date-1 price is P_M and F does not prepay any of L at date 0;
  - If $[\phi_1(b_1 + x_1) - b_0]P_M \leq L \leq [\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M)$ then the date-0 price is $L/[\phi_1(b_1 + x_1) - b_0]$ and F does not prepay any of L at date 0;
  - If $[\phi_1(b_1 + x_1) - b_0](P_M + \alpha_M) \leq L \leq \phi_1(b_1 + x_1)(P_M + \alpha_M)$ then the date-1 price is $P_M + \alpha_M$ and F prepays $L/(P_M + \alpha_M) - \phi_1(b_1 + x_1)$ at date 0.

A.3 Proof of Proposition 5

$\mathcal{P}_s$ is a convex set Let us show that the set of initial prices is convex. Let $P_0 > P_0'$ two feasible price levels associated with price level paths $\{P_t\}_{t\geq 0}$ and $\{P_t'\}_{t\geq 0}$. Let us show that any price level $P_0'' = \alpha P_0' + (1-\alpha)P_0$ with $\alpha \in [0,1]$ is also feasible. To this purpose, let us build a sequence of price levels $\{P_t''\}_{t\geq 0}$ so that $P_t'' = P_t'$ for any $t > 0$ and let us show that $\{P_t'', s_t\}_{t\geq 0}$ is a feasible policy.

Given this sequence of prices, we can use the budget constraint of the government to construct a sequence of nominal debt D_{t+1}'' because $b \mapsto b\phi_t(b)$ is continuously increasing. We find that D_t''/P_{t+1}'' is smaller than D_t'/P_{t+1}' but greater than D_t/P_{t+1} for any date $t \geq 0$.

Let us now show that the transversality condition is verified. For any $k > 0$, the government budget constraint leads to:

$$\bar{\phi}''_k \frac{D''_k/P''_{k+1}}{D''_{k-1}/P''_k} = \frac{\bar{\phi}''_k}{\phi''_k}\left(1 - \frac{s_k}{D''_{k-1}/P''_k}\right). \tag{58}$$

According to assumption 1, $\frac{\bar{\phi}''_k}{\phi''_k} \leq \frac{\bar{\phi}'_k}{\phi'_k}$ because the debt $D''_k \leq D'_k$, therefore:

$$\bar{\phi}''_k \frac{D''_k/P''_{k+1}}{D''_{k-1}/P''_k} \leq \frac{\bar{\phi}'_k}{\phi'_k}\left(1 - \frac{s_k}{D'_{k-1}/P'_k}\right) = \bar{\phi}'_k \frac{D'_k/P'_{k+1}}{D'_{k-1}/P'_k}. \tag{59}$$

The transversality condition can be rewritten as the limit when T tends to $+\infty$ of

$$\frac{D_{-1}}{P''_0} \prod_{k=0}^{T} \bar{\phi}''_k \frac{D''_k/P''_{k+1}}{D''_{k-1}/P''_k}. \tag{60}$$

Therefore inequality (59) shows that if the transversality condition is verified for the policy (P', s) it is also verified for (P'', s).

Given that the set of convex sets of real numbers are intervals, the set of initial prices is an interval of real numbers.

$\mathcal{P}_s$ and the game of chicken Suppose that $\mathcal{P}_s$ is not a singleton. Then it is immediate that we can find a policy $U' \in \mathcal{F}$ that violates (32) (with $s' = s$ and $P'_0 < P_0$).

Suppose now that the feasible policy U does not feature a game of chicken. There exists a feasible policy $U' \neq U$ in $\mathcal{F}$ that violates (32).

Either U' features $P'_0 < P_0$ and $s' \geq s$. We want to show that the policy $U'' = (P', s)$ is feasible $U'' \in \mathcal{F}$. Define $\{D'_t\}_{t\geq 0}$ and $\{D''_t\}_{t\geq 0}$ the associated paths of the debt. Because $b \mapsto b\phi_t(b)$ is continuously increasing, D''_t exists and is lower than D''_t. Besides the transversality condition is satisfied for the same argument than the one developed above (see equation (59)). Therefore $U'' \in \mathcal{F}$.

Or U' features $P'_0 = P_0$ and there exists $\tau \geq 0$ such that $s'_\tau < s_\tau$ and $s'_t \leq s_t$ otherwise. First, the same reasoning as above leads to prove that there exists a feasible policy U'' featuring $P'_0 = P_0$ and for $t = \tau$, $s'_\tau < s_\tau$ and $s'_t = s_t$ otherwise. Then, consider the problem at date $t = \tau$. If the legacy debt $B_{\tau-1}/P_\tau$ is the level of debt that leads to the maximal level of resources at date $\tau - 1$. In this case I THINK WE NEED STRICTLY INCREASING $b\phi(b)$ FUNCTION. Otherwise, one can raise surpluses at date τ to remain below s_τ and increase the surpluses at date $\tau - 1$. By doing so down to $t = 0$ we can prove that there exists a trajectory of surpluses such that $s''_0 < s_0$ and consistent with a feasible policy. We can convert this initial lower level of surplus into a lower level of price level and set all the surpluses equal to s_t as above. We thus prove that $\mathcal{P}_s$ is not a singleton.

A.4 Proof of Proposition 6

Rollover with diverging convenience yields implies pleasant monetary arithmetic Suppose that there exists a sequence $\{\tilde{b}_t\}_{t\geq 0}$ such that $\tilde{b}_{t-1} = \phi_t(\tilde{b}_t)\tilde{b}_t$ at any date $t \geq 0$ and $\sum_{t\geq 0} \delta(\tilde{b}_t)$ diverges.

Consider the policy U such that the price level is constant ($P_t = P$), $s_t = 0$ for any date $t > 0$; $s_0 = \frac{D_{-1}}{P}$; and $D_t = 0$ for any date $t \geq 0$. This policy with no public debt is always feasible as long as $s_0 < \bar{s}_0$ which is verified by choosing a sufficiently large price level P.

Consider now another policy U' such that $P'_t = P' = \frac{D_{-1}}{\tilde{b}_{-1}}$ and $s'_t = 0$ for any $t \geq 0$. The path of debt $\frac{D'_t}{P'} = \tilde{b}_t$ thus satisfies the budget constraint with $q_t = \phi_t(D'_t/P')$. Besides,

$$\prod_{i=t}^{\tau}\left(\bar{\phi}_i\left(\frac{D_i}{P_{i+1}}\right)\right)\frac{D_\tau}{P_{\tau+1}} = \prod_{i=t}^{\tau}\left(\frac{\bar{\phi}_i}{\phi_i}\right)\frac{D_{-1}}{P_0}, \tag{61}$$

$$ \tag{62}$$

Thus, the log of the product is simply

$$-\sum_{i=t}^{\tau}\delta_i + \ln\left(\frac{D_{-1}}{P_0}\right), \tag{63}$$

$$ \tag{64}$$

that tends to $-\infty$ which proves that the transversality condition is satisfied. Therefore U' is feasible. One can make sure that P' is lower than P because one can always increase P and reduces s_0, therefore the fiscal arithmetic is pleasant.

Pleasant monetary arithmetic implies the existence of a rollover with diverging convenience yields We prove this implication by contradiction.

Consider two feasible policies U and U' such that $U' \succ U$. We prove that $U' \neq U$ implies a contradiction.

For convenience, we denote by $b_t = B_t/P_{t+1}$, $Q_t = \prod_{i=0}^{t} q_i$ and $\bar{Q}_t = \prod_{i=0}^{t} \bar{q}_i$

First, since ϕ_t is decreasing, $b'_t \geq b_t$ for any t implies $q'_t \leq q_t$.

In addition, there exists a date $\tau \geq 0$ such that $b_{\tau'} > b_\tau$.

We compare the budget constraints for the two policies at any date $t \geq \tau$:

$$b'_t - b_t = s'_{t+1} - s_{t+1} + q'_{t+1}(b'_{t+1} - b_{t+1}) + (q'_{t+1} - q_{t+1})b_{t+1}, \tag{65}$$

which implies

$$b'_t - b_t \leq q'_{t+1}(b'_{t+1} - b_{t+1}). \tag{66}$$

Repeating this inequality from $t = \tau$ to an arbitrary $T > \tau$ we find:

$$q'_0 \dots q'_T b'_T \geq q'_0 \dots q'_T b_T + q'_0 \dots q'_\tau (b'_\tau - b_\tau). \tag{67}$$

The first right-hand-side member is positive and the second right-hand-side member is strictly positive and independent of T. Therefore, $Q'_T b'_T$ cannot converge to zero and,

$$Q'_T b'_T = \frac{Q'_T}{\bar{Q}'_T} \bar{Q}'_T b'_T \tag{68}$$

implies that $Q'_T / \bar{Q}'_T$ must be unbounded since $\bar{Q}'_T b'_T \to 0$.

Let now construct a rollover $\{\tilde{b}_t\}$. We define $\tilde{b}_0 = b'_0 - b_0$. Suppose that we can build a rollover up to date t such that $\tilde{b}_t \leq b'_t - b_t$. Inequality (66) combined with the facts that $(b'_{t+1} - b_{t+1}) < b'_{t+1}$ and that ϕ_t is decreasing show that we have:

$$b'_t - b_t \leq \phi_{t+1}(b'_{t+1} - b_{t+1})(b'_{t+1} - b_{t+1}), \tag{69}$$

which proves that there exists a feasible rollover at date $t+1$, $\tilde{b}_{t+1}$ such that $\tilde{b}_{t+1} \leq (b'_{t+1} - b_{t+1})$.

Finally, the fact that $Q'_T / \bar{Q}'_T$ must be unbounded means also that $\tilde{Q}'_T / \bar{\tilde{Q}}'_T$ must be unbounded. This is a contradiction since we assumed that any rollover must be such that $\sum_t \delta'_t$ converges.

Suppose there exists a policy profile $\{(P_t, s_t)\}_{t \geq 0}$ associated with the feasible policy U that does not feature a game of chicken. For instance, there exists another feasible policy U' featuring $P'_0 < P_0$, $P'_t = P_t$ otherwise and $s'_t = s_t$ for any $t \geq 0$. At date 0,

$$q_0 \left(\frac{B'_0}{P_1} \right) \frac{D'_0}{P_1} = \frac{D_{-1}}{P'_0} - s_0 > \frac{D_{-1}}{P_0} - s_0 = q_0 \left(\frac{D_0}{P_1} \right) \frac{D_0}{P_1}$$

Therefore, since the function $b \mapsto b\phi_0(b)$ is increasing, $D'_0 > D_0$. The same reasoning leads to $D'_t \geq D_t$ for any $t > 0$ (by induction). Therefore $U' \succ U$ and $U' \neq U$ which proves that the fiscal arithmetic is pleasant.

The exact same reasoning proves that if $P'_0 \leq P_0$ and $s'_t \leq s_t$ for any $t \geq 0$ with one strict inequality, then the fiscal arithmetic is pleasant.

Suppose now that the fiscal arithmetic is pleasant. Thus there exists $U' \succ U$ with $U' \neq U$ which contradicts the definition of the game of chicken 3.

A.5 Proof of Proposition 7

Properties of the lowest price level $\inf\{\mathcal{P}_s\}$ Consider two levels of initial debt: $D_{-1} < D'_{-1}$. Suppose that $\underline{\mathrm{P}}_s = \inf(\mathcal{P}_s(D_{-1})) \geq \underline{\mathrm{P}}'_s = \inf(\mathcal{P}_s(D'_{-1}))$. Suppose also that inf{
$mathcal P'_s\} > 0$. In addition, there exist a pair of positive scalars (ϵ_1, ϵ_2) such that $\underline{\mathrm{P}}_s + \epsilon_1 \in \mathcal{P}_s$ and $\underline{\mathrm{P}}'_s + \epsilon_2 \in \mathcal{P}'_s$ and $\underline{\mathrm{P}}_s + \epsilon_1 \geq \underline{\mathrm{P}}'_s + \epsilon_2$.

Let define $\{D_t\}_{t\geq 0}$ and $\{D'_t\}_{t\geq 0}$ the paths of debt associated with $(\underline{\mathrm{P}}_s + \epsilon_1, s)$ and D_{-1} and $(\underline{\mathrm{P}}'_s + \epsilon_2, s)$ and D'_{-1} respectively. Looking at the date-0 budget constraint we get:

$$\frac{D_{-1}}{\underline{\mathrm{P}}_s + \epsilon_1} = s_0 + b_0\phi_0(b_0), \tag{70}$$

$$\frac{D'_{-1}}{\underline{\mathrm{P}}'_s + \epsilon_2} = s_0 + b'_0\phi_0(b'_0). \tag{71}$$

So, if the initial debt is D_{-1} it is also possible to have a price P_0 such that:

$$\frac{D_{-1}}{P_0} = \frac{D'_{-1}}{\underline{\mathrm{P}}'_s + \epsilon_2} \text{ and hence } P_0 = (\underline{\mathrm{P}}'_s + \epsilon_2)\frac{D_{-1}}{D'_{-1}}, \tag{72}$$

$$\tag{73}$$

the real level of date-1 debt will simply be b'_0 which is feasible since it is consistent with $\{D'_t\}_{t\geq 0}$. Since ϵ_2 can be as small as we want, it proves that there exists a price level $P_0 \in \mathcal{P}_s$ that is below $\underline{\mathrm{P}}'_s$. This leads to a contradiction, thus $\inf(\mathcal{P}_s)$ strictly increases with D_{-1}.

A very similar argument proves that $\inf(\mathcal{P}_s)$ strictly decreases with s.

Rollover and bounded real level of debt We are going to prove that if there exists a level $\tilde{b} > 0$ such that there is no rollover $(b_t)_{t\geq 0}$ with $b_0 = \tilde{b}$, then the real level of debt is necessary bounded. The absence of rollover means that there exists $\tau > 0$ such that $b \mapsto b\phi_\tau(b)$ is bounded, let denote by M the upper bound. Therefore, the date-$\tau - 1$ real level of debt is bounded $D_{\tau-1}/P_\tau < \bar{s}_\tau + M$. By backward induction we can prove similarly that real levels of debt prior to date τ are also bounded. Finally, D_{-1}/P_0 is bounded proving that for any legacy debt $D_{-1} > 0$, the lowest feasible price $\inf\{\mathcal{P}_s\}$ is strictly positive.

A.6 Proof of Proposition 8

Consider a feasible policy (P, s). The existence of a rollover of reserves with diverging $\sum_{t\in\mathbb{N}} \delta_t^X(x_t)$ implies that the central bank can generate positive remittances for instance at date 0 and then rollover the reserves to infinity. This remittances add up to the revenue of the government and allows for reducing the initial price level or the surpluses at some point.

A.7 Proof of Proposition 9

If reserves and bonds are perfect substitute (equation (48)), then equation (49) is satisfied and we can apply all the results from section 4.2 replacing D by $\tilde{D}$; besides, since d does not appear in equation (49), the remittances and the central bank's balancesheet does not matter for the joint determination of (P, s).

More precisely, if the policy without remittances (P, s) is feasible for an initial legacy debt $\tilde{D}_{-1}$, then there exists a unique path of debt $\tilde{D}_t$ that satisfies the government budget constraint. The decomposition of this debt into (D, X, D^B) is irrelevant, the only relevant quantity being the overall net public liability $\tilde{D}_{-1}$. Reciprocally, if the extended policy (P, s, d) is feasible for an initial legacy debt $\tilde{D}_{-1}$ then the policy (P, s) is also feasible for a legacy debt $\tilde{D}_{-1}$.

As a consequence, we get $\mathcal{P}_s = \mathcal{P}_s^X$ for a given $\tilde{D}_{-1}$.

In addition, if the fiscal arithmetic is unpleasant then the set of feasible prices $\mathcal{P}_s$ for a given stream of surpluses s is simply a singleton –let call it $\{P_0\}$– and extending the policy space does not affect this set. Therefore, the fiscal arithmetic is also unpleasant for the extended policies and the unique feasible price level is determined by the stream of surpluses, that is, $\mathcal{P}_s = \mathcal{P}_s^X = \{P_0\}$.