Optimal growth, bequests and competitive equilibrium cycles in two-sector OLG models*

Elias CHAUMEIX

ENSAE
E-mail: elias.chaumeix@ensae-paristech.fr

Florian PELGRIN

EDHEC Business School
E-mail: florian.pelgrin@edhec.edu

and

Alain VENDITTI†‡

Aix-Marseille Univ., CNRS, EHESS, Centrale Marseille, AMSE
& EDHEC Business School
E-mail: alain.venditti@univ-amu.fr

This paper is dedicated to Pierre Cartigny (1946-2019) and Carine Nourry (1972-2019)

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*This work was supported by French National Research Agency Grants ANR-08-BLAN-0245-01 and ANR-17-EURE-0020. We thank W. Briec, X. Raurich, T. Seegmuller and G. Sorger for useful comments and suggestions. This paper also benefited from presentations at the "18th Journées Louis-André Gérard-Varet, Aix-en-Provence, June 13-14, 2019, and at the "19th Annual SAET Conference", Ischia, June 30-July 6, 2019.

†Pierre Cartigny was my PhD advisor. I met him first when I was a student and we became friends during my PhD. He has been a wonderful teacher and co-author. I will greatly miss him. Carine Nourry was a very close friend. I met her in 1994 as a student during her Master and we started to work jointly during her PhD. I remember as if it were yesterday how we discussed complex dynamics in optimal growth models with heterogeneous agents in our shared office in Marseille. She was a distinguished scholar and a wonderful co-author. I will also greatly miss my dear friend.

‡Corresponding author: AMSE, 5 Bd Maurice Bourdet, 13205 Marseille Cedex 01, France

Abstract: *The objective of this paper is to provide a simple model that can explain the long-run fluctuations of the annual flow of inheritance as identified by Piketty (2011) for France and Atkinson (2018) for the UK. We consider a two-sector Barro-type (1974) OLG model with non-separable preferences and bequests. The local stability properties of the optimal path apprear to depend on preferences through the sign of the cross derivative of the utility function, and on technologies through the sign of the capital intensity difference across the two sectors. We show in a first part that, under the assumption of a non-strictly concave utility function, preference and technology mechanisms can be separated and lead, each of them, to the existence of period-two cycles if the life-cycle utility function has a positive cross derivative across periods, and /or the consumption good is more capital intensive than the investment good. In a second part, considering a strictly concave utility function, the preference and technology mechanisms are now combined and can lead to the existence of quasi-periodic cycles through a Hopf bifurcation if the life-cycle utility function has a positive cross derivative across periods AND the consumption good is more capital intensive than the investment good. We also show that all these results are compatible with positive bequests.*

Keywords: *Two-sector overlapping generations model, optimal growth, endogenous fluctuations, periodic and quasi-periodic cycles, altruism, bequest*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

1 Introduction

It has been recently proved by Piketty [13] that in a country like France the annual flow of inheritance was about 20–25% of national income between 1820 and 1910, down to less than 5% in 1950, and back up to about 15% by 2010. The following graph indeed shows a long-run cyclic behavior of inheritance flows.

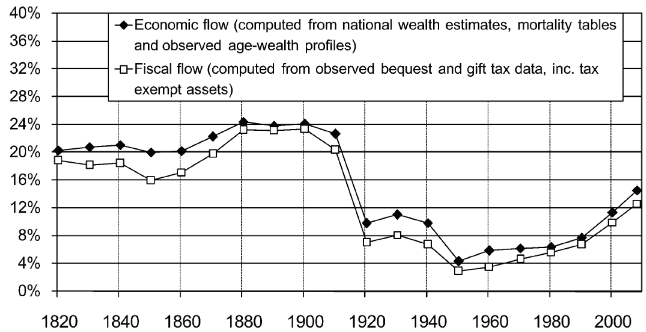


Figure 1: Annual inheritance flow as a fraction of national income, France 1820-2008 (Source: Piketty [13])

Similar conclusions have been reached by Atkinson [1] for the UK as shown in the following graph:

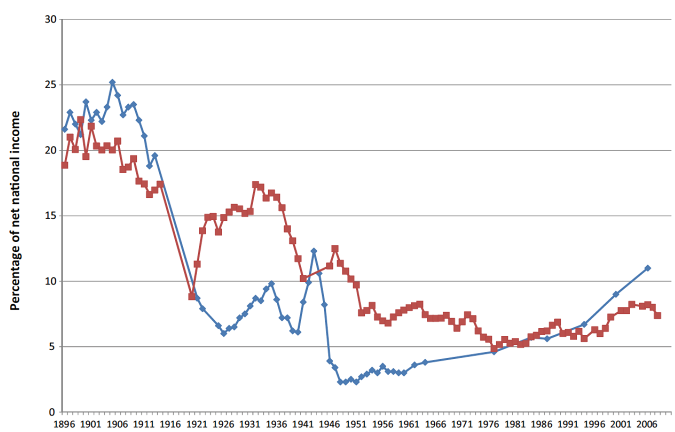


Figure 2: Comparison of France (red) and the United Kingdom (blue): transmitted wealth as percentage of net national income from 1896 to 2008 (Source: Atkinson [1])

The objective of this paper is to provide a simple model that can explain such long-run fluctuations. The standard model that allows to study inheritance flows across generations has been initially provided by Barro [3] with the concept of optimal bequest. As shown by Weil [17], as long as bequests are strictly positive across generations, the solution of the Barro model is equivalent to the solution of a Ramsey-type optimal growth model where a central planner maximizes the total intertemporal welfare.

Building on the well-known stability properties of the aggregate Ramsey model, it can be easily shown that if the life-cycle utility function of a representative generation living over two periods is additively separable, then the optimal path monotonically converges toward the steady state. In such a case there is no room for any cyclic behavior of bequests. But Michel and Venditti [12] have proved that if the life-cycle utility function is non-additively separable with a positive cross derivative across periods then endogenous period-two cycles can occur. This conclusion shows that such a model based on a preference mechanism is formally equivalent to a standard two-sector optimal growth model where period-two endogenous cycles rely on a technology mechanism and occur if the consumption good is more capital intensive than the investment good (see Benhabib and Nishimura [5]). Considering our goal to describe accurately the long run dynamics of bequests, the main critic of this result is that period-two cycles imply negative auto-correlations of variables which are not in line with the empirical properties of macroeconomic time series and bequests in particular.

The strategy in this paper is then to extend the Michel and Venditti [12] formulation to a two-sector economy. Beside introducing in the analysis both mechanisms relying on preference and technology, the extended model leads now to a dimension-four dynamical system which can give rise to the existence of quasi-periodic optimal paths, through the occurrence of complex characteristic roots, that are compatible with negative auto-correlation of variables and are in line with the long run empirical properties of aggregate time series. The analysis is divided in two parts. In a first part, under the assumption of a non-strictly concave utility function, we show that the preference and technology mechanisms can be separated and lead, each of them, to the existence of period-two cycles. The global dynamics can then be described as the product of two cycles implying complex properties of the optimal path. In a second part, considering a strictly concave utility function, the preference and technology mechanisms are now combined and can lead to the existence of quasi-periodic cycles through a Hopf bifurcation if the life-cycle utility function is non-additively separable with a positive cross derivative across periods and the consumption good is more capital intensive than the investment good. We also show that all these results are compatible with positive bequests.[1]

The paper is organized as follows. In Section 2 we present the two-sector model with non-additively separable preferences, define the optimal growth problem of the central planner, prove the existence of a steady state and derive the characteristic polynomial from which the stability analysis if conducted. The existence of period-two cycles under the assumption of a non-strictly concave utility function is discussed in Section 3 together with the presentation of a simple example to illustrate the main conditions. Section 4 contains the extension to the case of a strictly concave utility function. We provide general sufficient conditions that rule out the existence of complex characteristic roots and we consider a specific class of utility functions to prove the possible existence of a Hopf bifurcation and thus of quasi-periodic cycles. In Section 5 we show that all our previous conditions are compatible with the decentralized equilibrium characterized by strictly positive bequests. Concluding comments are provided in Section 6 and all the proofs are contained into a final Appendix.

2 The model

2.1 Production

We consider a two-sector economy with one pure consumption good y_0 and one capital good y. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0), \qquad y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$, the total amount of labor being normalized to 1.

Assumption 1. *Each production function* $f^i : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$, $i = 0, 1$, *is* C^2, *increasing in each argument, concave, homogeneous of degree one and such that for any* $x > 0$, $f^i_{k_i}(0, x) = f^i_{l_i}(x, 0) = +\infty$, $f^i_{k_i}(+\infty, x) = f^i_{l_i}(x, +\infty) = 0$.

For any given (k, y), we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

[1] Kalra [10] and Reichlin [14] provide conditions for the existence of Hopf cycles in two-sector OLG models but do not take into account bequests.

$$
\begin{aligned}
T(k,y) = \max_{k_0,k_1,l_0,l_1} \quad & f^0(k_0,l_0) \\
s.t. \quad & y \leq f^1(k_1,l_1) \\
& k_0 + k_1 \leq k \\
& l_0 + l_1 \leq 1 \\
& k_0, k_1, l_0, l_1 \geq 0
\end{aligned} \tag{1}
$$

The value function $T(k,y)$ is called the social production function and describes the frontier of the production possibility set. Constant returns to scale of technologies imply that $T(k,y)$ is concave non strictly. We will assume in the following that $T(k,y)$ is at least C^2.[2]

Denoting p the price of the investment good, r the rental rate of capital and w the wage rate, all in terms of the price of the consumption good, it is easy to show that

$$T_k(k,y,) = r(k,y), \;\; T_y(k,y) = -p(k,y) \tag{2}$$

and

$$w(k,y) = T(k,y) - r(k,y)k + p(k,y)y \tag{3}$$

We can also characterize the second derivatives of $T(k,y)$. From the concavity property we have:

$$T_{kk}(k,y) = \frac{\partial r}{\partial k} \leq 0, \;\; T_{yy}(k,y) = -\frac{\partial p}{\partial y} \leq 0$$

As shown by Benhabib and Nishimura [6], the sign of the cross derivative $T_{ky}(k,y)$ is given by the sign of the relative capital intensity difference between the two sectors. Denoting $a_{00} = l_0/y_0$, $a_{10} = k_0/y_0$, $a_{01} = l_1/y$ and $a_{11} = k_1/y$ the capital and labor coefficients in each sector, it is easy to derive from the constant returns to scale property that

$$\frac{dp}{dr} = a_{01}\left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}}\right) \equiv b \tag{4}$$

with b the relative capital intensity difference, and thus

$$T_{ky} \;\; = \;\; T_{yk} = -\frac{\partial p}{\partial r}\frac{\partial r}{\partial k} = -T_{kk}b$$

The sign of b and of T_{ky} is positive if and only if the investment good is capital intensive. Notice also that $T_{yy}(k,y)$ may be written as

$$T_{yy} \;\; = \;\; -\frac{\partial p}{\partial r}\frac{\partial r}{\partial y} = T_{kk}b^2$$

Remark: The derivative $dr/dp = b^{-1}$ is well-known in trade theory as the Stolper-Samuelson effect. Similarly, at constant prices, we can derive the associated Rybczinsky effect $dy/dk = b^{-1}$. We therefore find the well-known duality between the Rybczinsky and Stolper-Samuelson effects.

[2] A proof of the differentiability of $T(k,y)$ under Assumption 1 and non-joint production is provided in Benhabib and Nishimura [5].

2.2 Preferences

The economy is populated by a constant population of finitely-lived agents.[3] In each period t, $N_t = N$ persons are born, and they live for two periods: they work during the first (with one unit of labor supplied) and they have preferences for consumption (c_t, when they are young, and d_{t+1}, when they are old) which are summarized by the utility function $u(c_t, Bd_{t+1})$, with $B > 0$ a normalization constant, such that

Assumption 2. *$u(c, Bd)$ is increasing with respect to each argument ($u_1(c, Bd) > 0$ and $u_d(c, Bd) > 0$), concave and C^2 over the interior of $\mathbb{R}^2_+$. Moreover, $\lim_{X\to 0} Xu_X(c, X)/u_c(c, X) = 0$ and $\lim_{X\to +\infty} Xu_X(c, X)/u_c(c, X) = +\infty$, or $\lim_{X\to 0} Xu_X(c, X)/u_c(c, X) = +\infty$ and $\lim_{X\to +\infty} Xu_X(c, X)/u_c(C, X) = 0$.*

We also introduce a standard normality assumption between the two consumption levels

Assumption 3. *Consumptions c and d are normal goods.*

We finally introduce the following useful elasticities of substitution of consumptions:

$$\epsilon_{cc} = -u_c/u_{cc}c > 0, \qquad \epsilon_{cd} = -u_c/u_{cd}Bd, \tag{5}$$

$$\epsilon_{dc} = -u_d/u_{cd}c, \quad \epsilon_{dd} = -u_d/u_{dd}Bd > 0 \tag{6}$$

Notice that the normality Assumption 3 implies $1/\epsilon_{cc} - 1/\epsilon_{dc} \geq 0$ and $1/\epsilon_{dd} - 1/\epsilon_{cd} \geq 0$ and concavity in Assumption 2 implies $1/(\epsilon_{cc}\epsilon_{dd}) - 1/(\epsilon_{dc}\epsilon_{cd}) \geq 0$.

2.3 The optimal growth problem

Under complete depreciation within one period,[4] the capital accumulation equation is

$$k_{t+1} = y_t \tag{7}$$

Total labor being normalized to 1, we consider from now on that $N = 1$. At each time t total consumption is then given by the social production function, i.e. $c_t + d_t = T(k_t, y_t)$. The objective of the central planner combines utilities of successive generations

[3] An increasing population could be considered without altering all our results.

[4] Considering that in an OLG model one period is approximately 30 years, complete depreciation is a realistic assumption.

$$\max_{\{c_t, d_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(c_t, Bd_{t+1}) \tag{8}$$

where $\beta \in (0, 1]$ is the discount factor.[5] Considering (7) and the fact that $c_t = T(k_t, y_t) - d_t$, the optimization program (8) can be equivalently written as follows

$$\max_{\{d_{t+1}, k_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(T(k_t, k_{t+1}) - d_t, Bd_{t+1}) \tag{9}$$

with d_0 and k_0 given. The first order conditions are given by the following two difference equations of order two:

$$\begin{aligned} u_d(T(k_t, k_{t+1}) - d_t, Bd_{t+1})B - \beta u_c(T(k_{t+1}, k_{t+2}) - d_{t+1}, Bd_{t+2}) &= 0 \\ u_c(T(k_t, k_{t+1}) - d_t, Bd_{t+1})T_y(k_t, k_{t+1}) &+ \\ \beta u_c(T(k_{t+1}, k_{t+2}) - d_{t+1}, Bd_{t+2})T_k(k_{t+1}, k_{t+2}) &= 0 \end{aligned} \tag{10}$$

Any path from d_0 and k_0 given that satisfy equations (10) together with the following transversality conditions

$$\lim_{t \to +\infty} \beta^t u_d(c_t, Bd_{t+1}) p_{t+1} k_{t+1} = 0 \text{ and } \lim_{t \to +\infty} \beta^t u_d(c_t, Bd_{t+1}) d_{t+1} = 0$$

is an optimal path.

2.4 Steady state

A steady state is defined as $k_t = k^*$, $d_t = d^*$ for all t solutions of the following equations

$$\begin{aligned} \frac{u_d(T(k,k)-d, Bd)B}{u_c(T(k,k)-d, Bd)} &= \beta \\ -\frac{T_y(k,k)}{T_k(k,k)} &= \beta \end{aligned} \tag{11}$$

Beside discussing the existence and uniqueness of the steady state, we need also to use the normalization parameter B in order to normalize the stationary consumption d, rendering it constant when the discount factor β is modified. As in the standard two-sector model, we get the following result:

Proposition 1. *Under Assumptions 1-3, there exists a unique steady state* (k^*, d^*) *solution of equations* (11)*. Moreover, there exists a unique value* B^* *such when* $B = B^*$*, the stationary consumption* d^* *can be normalized to any value* $\bar{d} \in (0, T(k^*, k^*))$*.*

Proof. See Appendix 7.1. □

A pair (k^*, d^*) will be called the Modified Golden Rule. The stationary consumption of young agents is obtained from $c^* = T(k^*, k^*) - d^*$.

[5] In the case $\beta = 1$, the infinite sum into the optimization program (8) may not converge. In such a case we may apply the definition of optimality as provided by Ramsey [14].

2.5 Characteristic polynomial

Based on the above computations, the characteristic polynomial is derived from total differentiation of equations (10). Denoting $T(k^*, k^*) = T^*$, $T_k(k^*, k^*) = T^*_k$ and $T_{kk}(k^*, k^*) = T^*_{kk}$, let us define the following elasticities of the consumption good's output and the rental rate with respect to the capital stock, all evaluated at the steady state

$$\varepsilon_{ck} = T^*_k k^*/T^* > 0, \quad \varepsilon_{rk} = -T^*_{kk} k^*/T^*_k > 0 \tag{12}$$

We get:

Proposition 2. *Under Assumptions 1-3, the degree-4 characteristic polynomial is given by*

$$\mathcal{P}(\lambda) \;=\; \lambda^4 - \lambda^3 B + \lambda^2 C - \lambda \tfrac{B}{\beta} + \tfrac{1}{\beta^2} \tag{13}$$

with

$$\begin{aligned} B &= -\tfrac{\beta}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{\beta + b^2}{\beta b} + \tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \\ C &= -\tfrac{(1+\beta)}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{\beta + b^2}{\beta b} \left(\tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{2}{\beta} \end{aligned} \tag{14}$$

or equivalently

$$\begin{aligned} \mathcal{P}(\lambda) &= \left[\lambda^2 - \lambda \left(\tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{1}{\beta}\right] \tfrac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \\ &+ \lambda(\lambda - 1)\left(\lambda - \tfrac{1}{\beta}\right) \tfrac{\beta}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) \end{aligned} \tag{15}$$

If λ is a characteristic root of (15), then $\bar{\lambda}$, $(\beta\lambda)^{-1}$ and $(\beta\bar{\lambda})^{-1}$ are also characteristic roots of (15). Moreover, at least two roots or a pair of complex conjugate roots have a modulus larger than one, and one of the following cases necessarily hold:

i) the four roots are real and distincts,

ii) the four roots are given by two pairs of non-real complex conjugates,

iii) there are two complex conjugates double roots or two real double roots.

Proof. See Appendix 7.2. □

This Proposition is of crucial importance. It shows indeed that if there exist a pair of complex characteristic roots $(\lambda, \bar{\lambda})$ solutions of the quartic polynomial (15), then a second pair of complex characteristic roots as given by $(\beta\lambda)^{-1}$ and $(\beta\bar{\lambda})^{-1}$ are also solutions of (15). Therefore, Proposition 2 proves that the 4 characteristic roots are either all real or all complex. Proposition 2 also implies that at most two characteristic roots can have a modulus lower than 1 and thus that the steady state can be either saddle-point stable or totally unstable. Of course in this last case, periodic cycles can occur.

Remark 1: It important to notice that the degree-4 characteristic polynomial (13) is a quasi-palindromic equation that can be solved explicitly, and its roots can be determined using only quadratic equations (see Appendix 7.10 for details.).

Remark 2: Notice that if $b = 0$, we get the one-sector formulation with a two-dimensional dynamical system as considered in Michel and Venditti [12]. The characteristic polynomial can indeed be simplified as follows

$$\mathcal{P}(\lambda) \quad = \quad \lambda^2 - \lambda \frac{\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} - \frac{(1+\beta)}{\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}\right)}{1 - \frac{\beta}{\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}\right)} + \frac{1}{\beta}$$

The same conclusions as in Michel and Venditti [12] are obviously derived.

Similarly, if the utility function is additively separable, i.e. $u_{cd} = u_{dc} = 0$, we get the two-sector optimal growth formulation with a two-dimensional dynamical system as considered in Benhabib and Nishimura [5]. The characteristic polynomial can indeed be simplified as follows

$$\mathcal{P}(\lambda) \quad = \quad \lambda^2 - \lambda(1+\beta)\frac{\frac{\beta}{\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}} + (\beta + b^2)}{\frac{\beta}{\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}} + (1+\beta)b} + \frac{1}{\beta}$$

The same conclusions as in Benhabib and Nishimura [5] are then derived.

Under Assumption 2, the sign of the expression $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}$ is given by the sign of the cross derivative u_{cd}, i.e. by the opposite of the sign of $\epsilon_{cd}, \epsilon_{dc}$, which is a crucial ingredient to determine the local stability properties of the steady state. Moreover, we easily notice from (15) that if the utility function is non-strictly concave, i.e. if $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} = 0$, then the degree-4 polynomial simplifies to a product of two degree-2 polynomials which are then quite simple to solve. We then proceed in two steps, first focusing on the simpler case of a non-strictly concave utility function, and second considering the more general case of strictly concave preferences.

3 Period-two cycles under non-strictly concave preferences

Let us introduce the following Assumption:

Assumption 4. *The utility function $u(c, Bd)$ is concave non-strictly, i.e.* $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} = 0$.

As a preliminary result, we show that under such a restriction, the characteristic roots cannot be complex

Lemma 1. *Under Assumptions 1-4, the characteristic roots are real.*

Proof. See Appendix 7.3. □

Following simultaneously the same methodologies as in the two-sector optimal growth model and the optimal growth solution of the aggregate OLG model, we discuss the local stability properties of equilibrium paths depending both on the sign of the capital intensity difference across sectors b and the sign of the cross derivative u_{cd}, i.e. of the two elasticities ϵ_{cd} and ϵ_{dc}.

We first provide with the following Proposition some simple conditions ensuring the saddle-point property with monotone convergence.

Proposition 3. *Under Assumptions 1-4, if $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, then the equilibrium path is monotone and the steady-state (k^*, d^*) is a saddle-point.*

Proof. See Appendix 7.4. □

We now show that convergence with oscillations and persistent competitive equilibrium cycles may occur under a quite large set of circumstances.

Proposition 4. *Under Assumptions 1-4, the following results hold:*

i) When the investment good is capital intensive, i.e. $b \geq 0$, let $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$. Moreover, when ϵ_{cc} crosses the bifurcation values $-\epsilon_{dc}$ or $-\epsilon_{dc}/\beta$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

ii) When $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, let the consumption good be capital intensive, i.e. $b < 0$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $b \in (-\infty, -1) \cup (-\beta, 0)$. Moreover, if there is some $\beta^* \in (0, 1)$ such that $b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0, 1)$ such that, when β crosses $\bar{\beta}$ from above, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

iii) When the consumption good is capital intensive, i.e. $b < 0$, and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$, the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $b \in (-\infty, -1) \cup (-\beta, 0)$ and $\epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$. Moreover, if there is some $\beta^* \in (0, 1)$ such that $b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0, 1)$ such that, when β crosses $\bar{\beta}$ from above or ϵ_{cc} crosses the bifurcation values $-\epsilon_{dc}$ or $-\epsilon_{dc}/\beta$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

Proof. See Appendix 7.5. □

Proposition 4 provides two independent mechanisms leading to the existence of endogenous fluctuations. The first one is based on the properties of preferences through the sign of the cross derivative u_{cd} and is the more interesting as it allows to generate period-2 cycles in a two-sector model even under a capital intensive investment good sector, a condition which is known since Benhabib and Nishimura [6] to guarantee monotone convergence in a standard optimal growth model. In order to provide an economic intuition, let us consider an instantaneous increase in the capital stock k_t. From the equality $c_t + d_t = T(k_t, y_t)$ and the fact that $T_k > 0$, we derive that c_t increases, and thus, using the fact that the marginal utility of second period consumption u_d is larger as $u_{dc} > 0$, a constant utility level $u(c_t, d_{t+1})$ can be obtained from a decrease of d_{t+1}. Consider then the first equation in (10). We derive for a given d_{t+2}

$$\frac{\Delta c_{t+1}}{\Delta c_t} = \frac{u_{dc}}{u_{cc}\beta} + \frac{u_{dd}}{u_{cc}\beta}\frac{\Delta d_{t+1}}{\Delta c_t} < 0$$

It follows therefore from the equality $c_{t+1} + d_{t+1} = T(k_{t+1}, y_{t+1})$ that total consumption at time $t+1$ is lower, implying for a constant y_{t+1}, a lower capital stock k_{t+1}. Endogenous fluctuations are thus generated from consumption intertemporal allocations.

The second mechanism is, as in the two-sector optimal growth model, based on the properties of sectoral technologies through the sign of the capital intensity difference across sectors. Following Benhabib and Nishimura [6], we can use the Rybczinski and Stolper-Samuelson effects to provide a simple economic intuition for this result. Assume indeed that the consumption good is capital intensive, i.e. $b < 0$, and consider an instantaneous increase in the capital stock k_t. This results in two opposing forces:

- The trade-off in production becomes more favorable to the consumption good, and the Rybczinsky effect implies a decrease of the output of the capital good y_t. This tends to lower the investment and the capital stock in the next period k_{t+1}.

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. Indeed the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital and this tends to increase the investment and the capital stock in period $t+2$, k_{t+2}.

Of course, under both mechanisms, the existence of persistent fluctuations require that the oscillations in consumption and relative prices must not present intertemporal arbitrage opportunities. A minimum level of myopia, i.e. a low enough value for the discount rate β, is thus necessary. Note finally that in case iii) of Proposition 4, both mechanisms hold at the same time. It is important to mention here that this case is of particular interest. Indeed, using both β and ϵ_{cc} as two bifurcation parameters allows to consider a co-dimension 2 bifurcation which corresponds to the flip bifurcation with a 1:2 resonance where two characteristic roots are equal to -1 simultaneously. As shown in Kuznetsov [11], in such a configuration, under non-degeneracy conditions, the steady state is either saddle-point stable or elliptic. This last case may give rise to the existence of quasi-periodic cycles which are usually associated to a Hopf bifurcation.

Let us provide an illustration for all these cases assuming the particular class of homogeneous of degree $\gamma \leq 1$ utility functions with $B = 1$,[6] which obviously satisfies Assumptions 2 and 3. Building on this property, we introduce the share of first period consumption within total utility $\phi(c, d) \in (0, \gamma)$ defined as follows:

$$\phi(c, Bd) = \frac{u_c(c,Bd)c}{u(c,Bd)} \tag{16}$$

The share of second period consumption within total utility is similarly defined as $\gamma - \phi(c, Bd) \in (0, 1)$. From (5)-(6) we get

$$\epsilon_{cd} = -\frac{\epsilon_{cc}}{1-\epsilon_{cc}(1-\gamma)}, \quad \epsilon_{dc} = -\frac{(\gamma-\phi)\epsilon_{cc}}{\phi[1-\epsilon_{cc}(1-\gamma)]}, \quad \epsilon_{dd} = \frac{(\gamma-\phi)\epsilon_{cc}}{\phi-\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}$$

and concavity requires the following restriction:

Assumption 5. $\epsilon_{cc} < \frac{\gamma}{\phi(1-\gamma)}$

Under this restriction, we obviously get $\epsilon_{dd} > 0$ while $\epsilon_{cd}, \epsilon_{cd} < 0$ if and only if $\epsilon_{cc} < 1/(1-\gamma) \equiv \tilde{\epsilon}_{cc}(< \gamma/\phi(1-\gamma))$. Moreover, we can compute the elasticity of substitution between the two life-cycle consumption levels as

$$\sigma = \frac{\epsilon_{cc}(\gamma-\phi)}{\gamma-\phi\epsilon_{cc}(1-\gamma)}$$

We assume for now that $\gamma = 1$ which implies that Assumptions 4 and 5 hold. Let us focus first on the case i) of Proposition 4 where endogenous fluctuations arise under a capital intensive investment good. Both consumption levels are normal goods and the cross derivative u_{cd} is obviously positive.[7] We get the following Corollary:

[6] We do not need to introduce a normalization constant B with this class of utility function.

[7] These results are derived from concavity and standard Euler equalities for homogeneous functions, namely $u(c, Bd) = u_c(c, Bd)c + u_d(c, Bd)Bd$, $0 = u_{cc}(c, Bd)c + u_{cd}(c, Bd)Bd$ and $0 = u_{dc}(c, Bd)c + u_{dd}(c, Bd)Bd$.

Corollary 1. *Under Assumption 1, let the utility function $u(c, Bd)$ be linear homogeneous. Then, for any given sign of the capital intensity difference b, the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\phi \in (0, \underline{\phi}) \cup (\bar{\phi}, 1)$, with $\underline{\phi} = 1/2$ and $\bar{\phi} = 1/(1+\beta)$. Moreover, when ϕ crosses the bifurcation values $\underline{\phi}$ or $\bar{\phi}$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.*

Proof. See Appendix 7.6. □

Corollary 1 precisely illustrates the existence of period-2 cycles for a class of standard linear homogeneous preferences even when the investment good is capital intensive.

Let us now illustrate the case iii) of Proposition 4 where a co-dimension 2 bifurcation can arise. We need to consider precise specifications for the sectoral production functions. Assume as in Baierl *et al.* [2] that the consumption and investment goods are produced with Cobb-Douglas technologies as follows

$$y_0 = k_0^{\alpha_0} l_0^{1-\alpha_0}, \;\; y = k_1^{\alpha_1} l_1^{1-\alpha_1} \tag{17}$$

It can be shown that

$$b \;\; = \;\; \frac{\beta(\alpha_1 - \alpha_0)}{1-\alpha_0} \tag{18}$$

We then derive the following Corollary:

Corollary 2. *Let the utility function be homogeneous of degree 1 and the sectoral production functions be given by (17), and assume that $\alpha_0 > (1+\alpha_1)/2$ such that $b \in (-\infty, -1)$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\phi \in (0, \underline{\phi}) \cup (\bar{\phi}, 1)$ and $\beta > \underline{\beta}$, with $\underline{\phi} = 1/2$, $\bar{\phi} = 1/(1+\beta)$ and $\underline{\beta} = (1-\alpha_0)/(\alpha_0 - \alpha_1)$. If $\beta = \underline{\beta}$ and $\phi = \bar{\phi}$ or $\underline{\phi}$, then a co-dimension 2 flip bifurcation with a 1:2 resonance generically occurs.*

Proof. See Appendix 7.7. □

While providing a precise dynamic analysis of this co-dimension 2 bifurcation goes far beyond the goal of this paper, it is worthwhile to mention that this case provides an interesting possibility of smooth endogenous fluctuations for the main aggregate variables which does not arise under a standard flip bifurcation. Indeed, while there does not a priori exist complex characteristic roots under a linear homogenous utility function, Kuznetsov [11] shows that under a 1:2 resonance, the steady state can be elliptic and a stable limit cycle, similar to those that arise under a Hopf bifurcation, can occur. As we will show in the next section, a Hopf bifurcation provides a better tool to describe the long-run cyclical behavior of macroeconomic variables such as bequests.

4 Quasi-periodic cycles under strictly concave preferences

Up to now we have simplified the analysis to the consideration of a non-strictly concave utility function in order to reduce the degree-4 characteristic polynomial to the product of two degree-2 polynomials. In such a framework, we have shown that the characteristic roots are necessarily real and that endogenous fluctuations can occur through the existence of period-two cycles. But from an empirical point of view, period-two cycles are associated to the unrealistic property of negative auto-correlation of variables. In order to solve this problem, we need to focus on the existence of complex characteristic roots for which quasi-periodic cycles occurring through a Hopf bifurcation can generate fluctuations that are compatible with positive auto-correlations. Such a property is required to provide an empirically relevant description of smooth long-run fluctuations of variables such as bequests.

We can start by providing general sufficient conditions allowing to rule out the existence of complex roots.

Proposition 5. *Under Assumptions 1-3, let the utility function* $u(c, Bd)$ *be strictly concave. Then the roots of the characteristic polynomial (15) are necessarily real in the following cases:*

i) for any sign of $\epsilon_{cd}, \epsilon_{cd}$ *if the investment good sector is capital intensive, i.e.* $b > 0$*,*

ii) if $\epsilon_{cd}, \epsilon_{cd} > 0$ *and the consumption good sector is capital intensive, i.e.* $b < 0$*.*

Proof. See Appendix 7.8. □

Necessary conditions for the existence of complex roots are therefore based on the two mechanisms that generate endogenous fluctuations in the non-strictly concave case, namely $b < 0$ **and** $\epsilon_{cd}, \epsilon_{cd} < 0$. In order to study whether complex characteristic roots and a Hopf bifurcation with quasi-periodic cycles can occur, let us consider again a utility function homogeneous of degree γ, but now assuming $\gamma < 1$ to allow for strict concavity.

We first provide sufficient conditions to ensure saddle-point property of the steady state with real characteristic roots.

Proposition 6. *Let the utility function be homogeneous of degree* $\gamma < 1$*, and assume that* $\epsilon_{cc} < \tilde{\epsilon}_{cc}$*,* $b \in (-\infty, -1) \cup (-\beta, 0)$ *and*

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} > 1 \tag{19}$$

Then there exist $0 < \underline{\phi} \leq \bar{\phi} < \gamma$ *and* $\hat{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$ *such that when* $\phi \in (0, \underline{\phi}) \cup (\bar{\phi}, \gamma)$ *the characteristic roots are real and the steady-state is saddle-point stable. Moreover,*

i) when $\phi \in (\bar{\phi}, \gamma)$*, the optimal path converges towards the steady state with oscillations if* $\epsilon_{cc} \in (0, \hat{\epsilon}_{cc})$ *or monotonically if* $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$*,*

ii) when $\phi \in (0, \underline{\phi})$*, the optimal path converges towards the steady state with oscillations.*

Proof. See Appendix 7.9. □

Condition (19) allows to get the existence of the bound $\hat{\epsilon}_{cc}$ and thus of the existence of oscillations when $\phi \in (\bar{\phi}, \gamma)$. This restriction may be easily interpreted. Denoting σ_i the elasticity of capital-labor substitution in sector $i = 0, 1$ and using Drugeon [9], we may relate the ratio of elasticities $\varepsilon_{ck}/\varepsilon_{rk}$ to an aggregate elasticity of substitution between capital and labor, denoted Σ, which is obtained as a weighted sum of the sectoral elasticities σ_i. We have indeed:[8]

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}} = \left(\frac{T}{l_0^2}\right) \frac{s}{1-s} \frac{\Sigma}{GDP} \text{ with } \Sigma = \frac{GDP}{pykT} \left(pyk_0 l_0 \sigma_0 + Tk_1 l_1 \sigma_1\right) \tag{20}$$

$GDP = T + py$ and $s = rk/GDP$ the share of capital income in GDP. Therefore, oscillations when $\phi \in (\bar{\phi}, \gamma)$ are associated with a large aggregate elasticity of substitution between capital and labor *i.e.*, large enough sectoral elasticities of capital-labor substitution.

Proposition 6 implies that the existence of complex roots, if any, requires to consider values of ϕ such that $\phi \in (\underline{\phi}, \bar{\phi})$. We can then derive the following result that provides sufficient conditions for the occurrence of a Hopf bifurcation:

Proposition 7. *Let the utility function be homogeneous of degree* $\gamma < 1$*, and assume that* $\epsilon_{cc} < \tilde{\epsilon}_{cc}$ *and* $b \in (-\beta, 0)$*. Then there exist* $\bar{b} \in (-\beta, 1)$*,* $\underline{\gamma} \in (0, 1)$*,* $\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$*,* $\bar{\varepsilon} > 0$ *and four critical values* $(\underline{\phi} \leq)\underline{\phi}^c < \underline{\phi}^H < \bar{\phi}^H < \bar{\phi}^c(\leq \bar{\phi})$ *such that when* $b \in (-\beta, \bar{b})$*,* $\gamma \in (\underline{\gamma}, 1)$*,* $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc})$ *and*

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \bar{\varepsilon} \tag{21}$$

the following results hold:

[8] The expression of Σ is derived from Proposition 2 in Drugeon [9].

i) the steady state (k^*, d^*) is saddle-point stable with damped oscillations if $\phi \in (\underline{\phi}^c, \underline{\phi}^H) \cup (\bar{\phi}^H, \bar{\phi}^c)$,

ii) when ϕ crosses the bifurcation values $\underline{\phi}^H$ or $\bar{\phi}^H$, (k^*, d^*) undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles.

Proof. See Appendix 7.10. □

From a theoretical point of view, Proposition 7 provides a strong conclusion as it shows that a Hopf bifurcation and quasi-periodic cycles can occur in a two-sector optimal growth framework as long as it is based on an OLG structure with non-separable and strictly concave preferences. More specifically, we need intermediate values for the elasticity of intertemporal substitution in consumption and, using (20), not too large values for the sectoral elasticities of capital-labor substitution.

Such a result is drastically different from what can be obtained in standard optimal growth models as the existence of complex roots requires to consider at least three sectors.[9] From an empirical point of view, Proposition 7 also provides a strong conclusion which is related to the quasi periodicity of the cycles leading to positive auto-correlations of variables. Such a property is required to provide an empirically relevant description of long-run fluctuations of bequests.

Let us now focus on a numerical illustration. Considering that the annual discount factor is often estimated to be around 0.96 and that one period in an OLG model is about 30 years, we consider here that $\beta = 0.96^{30} \approx 0.294$. Focusing on a slight deviation with respect to the linear homogeneous case with $\gamma = 0.98$, let us then assume a standard value $\epsilon_{cc} = 1$ that satisfies $\epsilon_{cc} < \bar{\epsilon}_{cc}$. We also consider sectoral Cobb-Douglas technologies as given by (17) with $\alpha_0 = 0.6$ and $\alpha_1 = 0.21$ so that the consumption good is capital intensive with $b \approx -0.28665$ close to $-\beta$. The bounds exhibited in Proposition 7 are equal to $\underline{\phi}^c \approx 0.38858$ and $\bar{\phi}^c \approx 0.865$. We then find that the characteristic polynomial (49) admits four characteristic roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that are complex conjugate by pair with $\lambda_1\lambda_2 > 1$ and $\lambda_3\lambda_4 < 1$ if $\phi \in (\underline{\phi}, \underline{\phi}^H) \cup (\bar{\phi}^H, \bar{\phi})$ while $\lambda_3\lambda_4 > 1$ if $\phi \in (\underline{\phi}^H, \bar{\phi}^H)$, with $\underline{\phi}^H \equiv 0.5674$ and $\bar{\phi}^H \equiv 0.6713$. Moreover $\lambda_3\lambda_4 = 1$ when $\phi = \underline{\phi}^H$ or $\bar{\phi}^H$. As a result $\underline{\phi}^H$ and $\bar{\phi}^H$ are Hopf bifurcation values giving rise to quasi-periodic cycles in their neighborhood.

We need finally to show that the existence of optimal endogenous cycles is compatible with strictly positive bequest transmissions across generations.

[9] See Benhabib and Nishimura [5], Cartigny and Venditti [7], Venditti [16].

5 The solution with altruistic agents and a bequest motive

Let us consider a decentralized economy composed of overlapping generations of parents loving their children. As in the Barro [3] formulation, each agent is altruistic towards his descendant through a bequest motive. Parents indeed care about their child's welfare by taking into account their child's utility into their own utility function. They are now price-takers, considering as given the prices p_t, w_t and r_{t+1} as defined by (2) and (3), and determine their optimal decisions with respect to their budget constraints

$$w_t + p_t x_t = c_t + s_t \text{ and } R_{t+1} s_t = d_{t+1} + p_{t+1} x_{t+1} \tag{22}$$

with $R_{t+1} = r_{t+1}/p_t$ the gross rate of return, s_t the savings of young agents born in t and x_t the amount of bequest transmitted at time t by agents born in $t-1$. Note that bequest x_t is expressed as an investment good and requires the relative price p_t to enter the budget constraints. In each period, bequests must be non-negative:

$$x_t \geq 0 \text{ for all } t \geq 0 \tag{23}$$

An altruistic agent has a utility function given by the following Bellman equation

$$\begin{aligned} V_t(x_t) &= \max_{c_t, d_{t+1}, s_t, x_{t+1}} \{u(c_t, Bd_{t+1}) + \beta V_{t+1}(x_{t+1})\} \\ &= \max_{\{c_t, d_{t+1}, s_t, x_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(c_t, Bd_{t+1}) \end{aligned} \tag{24}$$

subject to (22) and (23). β is now interpreted as the intergenerational degree of altruism. It is well-known from the first welfare theorem that this altruistic problem is equivalent to the central planner problem (8), and the equilibrium is the unique Pareto optimum which coincides with the centralized solution. However, such an equivalence requires the non-negativity constraints of bequests (23) to hold with a strict inequality in order to preserve the link across generations.

Denoting q_t the shadow price of bequest x_t, we define the generalized Lagrangian associated to the optimization program (24)

$$\mathcal{L} = u(c_t, Bd_{t+1}) + \beta \frac{q_{t+1}}{p_{t+1}} \left[R_{t+1}(w_t + p_t x_t - c_t) - d_{t+1} \right] - q_t x_t$$

The first order conditions are the following

$$
\begin{aligned}
u_c(c_t, Bd_{t+1}) &= \tfrac{\beta q_{t+1} R_{t+1}}{p_{t+1}} \\
u_d(c_t, Bd_{t+1})B &= \tfrac{\beta q_{t+1}}{p_{t+1}} \\
\tfrac{\beta q_{t+1} R_{t+1} p_t}{p_{t+1}} &\leq q_t \text{ with an equality if } x_t > 0
\end{aligned}
$$

Consider now the two budget constraints in (22) evaluated at the steady state. Solving with respect to s_t using the fact that $s_t = p_t y_t = p_t k_{t+1}$ and $R_{t+1} = r_{t+1}/p_t$ we get

$$
\begin{aligned}
p^* x^* \left(1 - \tfrac{1}{R^*}\right) &= c^* + \tfrac{d^*}{R^*} - w^* = T(k^*, k^*) - w^* - d^* \left(1 - \tfrac{1}{R^*}\right) \\
&= (r^* k^* - d^*) \left(1 - \tfrac{1}{R^*}\right)
\end{aligned}
\tag{25}
$$

If $x^* > 0$, i.e. $r^* k^* > d^*$, then we derive from the fist order conditions that $R^* = r^*/p^* = \beta^{-1}$ and $u_d(c^*, Bd^*) = \beta u_c(c^*, Bd^*)$, which are exactly the same conditions as (11). We then obtain:

Proposition 8. *Under Assumptions 1-3, for any $\beta \in (0,1)$, there exists a unique value B^* such that when $B = B^*$, bequests are positive in the economy with degree of altruism equal to β.*

Proof. See Appendix 7.11. □

When bequests are positive at the steady state, then by continuity there are positive in a neighborhood of the steady state and the local stability properties provided in Propositions 3, 4 and 7 hold. In particular, the existence of optimal cycles and business fluctuations hold under positive bequests.

In order to illustrate this result, let us consider first the linear homogeneous utility function previously considered with $\gamma = 1$, and the Cobb-Douglas production structure as given by (17). Using (46) in Appendix 7.7, we derive that $r^* k^* > d^*$ and thus $x^* > 0$ if and only if

$$
\tfrac{\alpha_0 \beta \phi}{1-\phi} - (1 - \alpha_0 - \beta \alpha_1) > 0
$$

It follows immediately that if $\alpha_1 > 1 - \alpha_0$ and $\beta > (1 - \alpha_0)/\alpha_1$, then $1 - \alpha_0 - \beta \alpha_1 < 0$ and $x^* > 0$ for any $\phi \in (0,1)$. The existence of periodic cycles is thus compatible with positive bequests. Similarly, when $\alpha_1 < 1 - \alpha_0$, straightforward computations show that $x^* > 0$ if and only if

$$
\phi \tfrac{\gamma(1-\alpha_0-\beta\alpha_1)}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \equiv \tilde{\phi}_1
$$

It follows that the conditions of Corollary 1 for the existence of period-2 cycles can be satisfied if $\tilde{\phi} < \underline{\phi} = 1/2$. Sufficient conditions for this inequality to be satisfied are given by $\alpha_1 \in (1 - 2\alpha_0, 1 - \alpha_0)$ and $\beta >$

$(1-\alpha_0)/(\alpha_0+\alpha_1) \equiv \underline{\beta}$ with $\underline{\beta} < 1$. This example clearly shows that when the degree of altruism is large enough, endogenous optimal fluctuations are compatible with positive bequests. Moreover, this result holds for any sign of the capital intensity difference across sectors.

It is worth noticing that if, under $\alpha_1 \in (1-2\alpha_0, 1-\alpha_0)$, we assume that $\phi > \tilde{\phi}$ with $\tilde{\phi} > \bar{\phi}$, then bequests are positive but the conditions of Corollary 1 for the existence of period-2 cycles cannot be satisfied and the steady state is saddle-point stable. This inequality is satisfied if and only if $\alpha_1 \in (1-2\alpha_0, 1-\alpha_0)$, $\alpha_0 < 1/2$ and $\beta < (1-2\alpha_0)/\alpha_1$. Therefore, if the degree of altruism is not large enough, persistent endogenous fluctuations cannot arise.

Let us finally illustrate the possible existence of quasi-periodic cycles under positive bequests when the utility function is homogeneous of degree $\gamma < 1$ as in Section 4. Using again (46) in Appendix 7.7, we derive that $r^*k^* > d^*$ and thus $x^* > 0$ if and only if

$$\alpha_0\phi\beta - (\gamma-\phi)(1-\alpha_0-\beta\alpha_1) > 0$$

Consider then the particular illustration in Section 4 which is such that $1-\alpha_0-\beta\alpha_1 > 0$ and $\alpha_0 > \alpha_1$. It follows that bequests are positive if and only if

$$\phi > \frac{\gamma(1-\alpha_0-\beta\alpha_1)}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \equiv \tilde{\phi}_\gamma$$

With $\gamma = 0.98$, $\alpha_0 = 0.6$ and $\alpha_1 = 0.21$, we get $\tilde{\phi}_\gamma \approx 0.644 \in (\underline{\phi}^H, \bar{\phi}^H)$. It follows that positive bequests are compatible with quasi-periodic cycles. Indeed, the steady state, which is characterized by strictly positive bequests if $\phi > \tilde{\phi}_\gamma$, is saddle-point stable with damped oscillations if and only if $\phi \in (\bar{\phi}^H, \bar{\phi})$. Moreover, when ϕ crosses the bifurcation values $\bar{\phi}^H$ from above, the steady state undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles and thus long-run fluctuations of bequests.

6 Concluding comments

7 Appendix

7.1 Proof of Proposition 1

Consider in a first step the second equation in (11). Notice that the steady state value for k only depends on the characteristics of the technologies and is independent from the utility function. Moreover, this equation is equivalent to the equation which defines the stationary capital stock of a standard two-sector optimal growth model. The proof of Theorem 3.1 in Becker and Tsyganov [4] restricted to the case of one homogeneous agent applies so that there exists one unique k^* solution of this equation.

Consider now the first equation in (11) evaluated at k^*. We get:

$$\frac{u_d(T(k^*,k^*)-d,Bd)B}{u_c(T(k^*,k^*)-d,Bd)} \equiv h(d) \quad = \quad \beta \tag{26}$$

The function $h(d)$ is defined over $(0, T(k^*, k^*))$ and satisfies

$$h'(d) = \frac{\frac{Bu_{dd}}{u_d} - \frac{u_{cd}}{u_c} + \frac{u_{cc}}{u_c} - \frac{Bu_{cd}}{u_d}}{u_c u_d} = -\beta\left[\frac{1}{d}\left(\frac{1}{\epsilon_{dd}} - \frac{1}{\epsilon_{cd}}\right) + \frac{1}{c}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}}\right)\right]$$

Assumption 3 implies that $h'(d) < 0$. This monotonicity property together with the boundary conditions in Assumption 2 finally ensure the existence and uniqueness of a solution $d^* \in (0, T(k^*, k^*))$ of equation (26).

For a given k^*, consider a particular value $d^* = \bar{d} \in (0, T(k^*, k^*))$. $\bar{d}$ is a steady state if

$$\frac{u_d(T(k^*,k^*)-\bar{d},B\bar{d})B}{u_c(T(k^*,k^*)-d,Bd)} \equiv g(B) \quad = \quad \beta \tag{27}$$

We easily get

$$g'(B) \quad = \quad -\frac{u_d}{u_c}\left[\frac{1}{\epsilon_{dd}} - \frac{1}{\epsilon_{cd}} - 1\right]$$

which is generically different from zero. Therefore, under the boundary conditions in Assumption 2, there generically exists a unique value B^* such that when $B = B^*$, $d^* = \bar{d}$ is a normalized steady state.

□

7.2 Proof of Proposition 2

Using (5)-(6) and the fact that at the steady state $-T_y^* = \beta T_k^*$, total differentiation of the first order equations (10) gives after tedious but straightforward computations:

$$\begin{aligned}
& -\Delta k_t \frac{\beta T_k^* \epsilon_{cc}}{\epsilon_{dc}} + \Delta k_{t+1}\beta T_k^*\left(1 + \frac{\beta\epsilon_{cc}}{\epsilon_{dc}}\right) + \Delta d_t \frac{\beta\epsilon_{cc}}{\epsilon_{dc}} - \Delta d_{t+1}\beta\left(1 + \frac{\beta\epsilon_{cc}\epsilon_{cd}}{\epsilon_{dc}\epsilon_{dd}}\right) \\
= \quad & \Delta k_{t+2}\beta^2 T_k^* - \Delta d_{t+2}\frac{\beta^2\epsilon_{cc}}{\epsilon_{dc}} \\
& \Delta k_t\left(\frac{\beta T_k^{*2}}{\epsilon_{cc}c^* T_{kk}^*} - b\right) - \Delta k_{t+1}\left(\frac{\beta(1+\beta)T_k^{*2}}{\epsilon_{cc}c^* T_{kk}^*} - \Delta - b^2\right) - \Delta d_t \frac{\beta T_k^*}{\epsilon_{cc}c^* T_{kk}^*} \\
+ \quad & \Delta d_{t+1}\frac{\beta T_k^*}{\epsilon_{cc}c^* T_{kk}^*}\left(1 + \frac{\beta\epsilon_{cc}}{\epsilon_{dc}}\right) = -\Delta k_{t+2}\beta\left(\frac{\beta T_k^{*2}}{\epsilon_{cc}c^* T_{kk}^*} - b\right) + \Delta d_{t+2}\frac{\beta^2 T_k^*}{\epsilon_{cc}c^* T_{kk}^*}
\end{aligned}$$

Denoting $\Delta\xi_t = \Delta k_{t+1}$ and $\Delta\zeta_t = \Delta d_{t+1}$, we get the following matrix expression of the previous linear system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta^2 T_k^* & -\frac{\beta^2\epsilon_{cc}}{\epsilon_{dc}} \\ 0 & 0 & -\left(\frac{\beta T_k^{*2}}{\epsilon_{cc}c^* T_{kk}^*} - b\right) & \frac{\beta^2 T_k^*}{\epsilon_{cc}c^* T_{kk}^*} \end{pmatrix} \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \xi_{t+1} \\ \Delta \zeta_{t+1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\beta T_k^*\epsilon_{cc}}{\epsilon_{dc}} & \frac{\beta\epsilon_{cc}}{\epsilon_{dc}} & \beta T_k^*\left(1 + \frac{\beta\epsilon_{cc}}{\epsilon_{dc}}\right) & -\beta\left(1 + \frac{\beta\epsilon_{cc}\epsilon_{cd}}{\epsilon_{dc}\epsilon_{dd}}\right) \\ \frac{\beta T_k^{*2}}{\epsilon_{cc}c^* T_{kk}^*} - b & \frac{\beta T_k^*}{\epsilon_{cc}c^* T_{kk}^*} & -\frac{\beta(1+\beta)T_k^{*2}}{\epsilon_{cc}c^* T_{kk}^*} + \beta + b^2 & \frac{\beta T_k^*}{\epsilon_{cc}c^* T_{kk}^*}\left(1 + \frac{\beta\epsilon_{cc}}{\epsilon_{dc}}\right) \end{pmatrix} \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix}$$

$$\Leftrightarrow \quad A \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \xi_{t+1} \\ \Delta \zeta_{t+1} \end{pmatrix} = B \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix}$$

with

$$A = \begin{pmatrix} I & 0 \\ 0 & A_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & I \\ B_{21} & B_{22} \end{pmatrix}$$

Matrix A is invertible as $\det A = \det A_{22} = \delta^3 b\epsilon_{cc}/\epsilon_{dc}$, and we get

$$A^{-1} = \begin{pmatrix} I & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \text{ with } A_{22}^{-1} = \begin{pmatrix} \frac{T_k^*}{\beta b\epsilon_{cc}c^* T_{kk}^*} & \frac{1}{\beta b} \\ \frac{\epsilon_{dc}}{\beta^2\epsilon_{cc}}\left(\frac{\beta T_k^{*2}}{b\epsilon_{cc}c^* T_{kk}^*} - 1\right) & \frac{\epsilon_{dc}T_k^*}{\beta b\epsilon_{cc}} \end{pmatrix}$$

The linearized dynamical system can then be expressed as follows

$$\begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \xi_{t+1} \\ \Delta \zeta_{t+1} \end{pmatrix} = A^{-1}B \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_{22}^{-1}B_{21} & A_{22}^{-1}B_{22} \end{pmatrix} \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix} \equiv J \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix}$$

Using (12), tedious but straightforward computations give the characteristic polynomial

$$\mathcal{P}(\lambda) \quad = \quad \lambda^4 - \lambda^3 B + \lambda^2 C - \lambda\frac{B}{\beta} + \frac{1}{\beta^2} \tag{28}$$

with

$$\begin{aligned} B &= -\frac{\beta}{b\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{\beta+b^2}{\beta b} + \frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \\ C &= -\frac{(1+\beta)}{b\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{\beta+b^2}{\beta b}\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{2}{\beta} \end{aligned} \tag{29}$$

After simplifications we get the expression (15).

Consider now that λ is a root of the characteristic polynomial (15), i.e. $\mathcal{P}(\lambda) = 0$. It follows obviously that if λ is complex then its conjugate $\bar{\lambda}$ is also a characteristic root. Let us then consider $\mathcal{P}((\beta\lambda)^{-1})$, namely

$$\begin{aligned}
\mathcal{P}\left(\frac{1}{\beta\lambda}\right) &= \left[\frac{1}{\beta^2\lambda^2} - \frac{1}{\beta\lambda}\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{1}{\beta}\right] \frac{\left(\frac{b}{\beta\lambda}-1\right)\left(\frac{1}{\lambda}-b\right))}{\beta b} \\
&+ \frac{1}{\beta\lambda}\left(\frac{1}{\beta\lambda}-1\right)\left(\frac{1}{\beta\lambda}-\frac{1}{\beta}\right)\frac{\beta}{b\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}}-\frac{\epsilon_{cd}}{\epsilon_{dd}}\right) \\
&= \frac{1}{\beta^4\lambda^4}\Bigg\{\left[\lambda^2 - \lambda\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{1}{\beta}\right]\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\
&+ \lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right)\frac{\beta}{b\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}}-\frac{\epsilon_{cd}}{\epsilon_{dd}}\right)\Bigg\} = 0
\end{aligned}$$

If follows that $(\beta\lambda)^{-1}$ is also a characteristic root. The same argument applies for $(\beta\bar{\lambda})^{-1}$. It follows that the four characteristic roots are either all real, or given by two pairs of complex conjugates. Moreover, at least two roots or a pair of complex conjugate roots have a modulus larger than one.

The nature of the characteristic roots can be derived considering the following expressions:

$$\begin{aligned}
\Delta &= \frac{256}{\beta^6} - \frac{192B^2}{\beta^5} - \frac{128C^2}{\beta^4} + \frac{288B^2C}{\beta^4} - \frac{60B^4}{\beta^4} - \frac{80B^2C^2}{\beta^3} + \frac{36B^4C}{\beta^3} \\
&- \frac{4B^6}{\beta^3} + \frac{16C^4}{\beta^2} - \frac{8B^2C^3}{\beta^2} + \frac{B^4C^2}{\beta^2} \\
D &= \frac{64}{\beta^2} - 16C^2 + 16B^2C - \frac{16B^2}{\beta} - 3B^4 \\
P &= 8C - 3B^2 \\
R &= B\left[B^2 + \frac{8}{\beta} - 4C\right]
\end{aligned} \tag{30}$$

Since we already know that the characteristic roots are either all real, or all complex, we immediately derive that $\Delta \geq 0$. Tedious but straightforward computations also show that

$$\begin{aligned}
D &= \frac{R}{B}\left[\frac{8}{\beta} - 3B^2 + 4C\right] \\
\Delta &= \frac{\left(\beta^2C^2 - 4\beta B^2 + 4\beta C + 4\right)R^2}{\beta^4B^2}
\end{aligned} \tag{31}$$

It follows that if $R = 0$ then $D = 0$ and $\Delta = 0$. This implies the following characterization of the roots:

i) when $\Delta > 0$ the characteristic roots are real and distincts if $P < 0$ and $D < 0$, and given by two pairs of non-real complex conjugates if $P > 0$ or $D > 0$;

ii) when $\Delta = R = D = 0$, there are two complex conjugates double roots or two real double roots depending on whether $P > 0$ or $P < 0$.

□

7.3 Proof of Lemma 1

Under Assumption 4, let us denote the two degree-2 polynomials as follows

$$\mathcal{P}_1(\lambda) = \lambda^2 - \lambda\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{1}{\beta}, \quad \mathcal{P}_2(\lambda) = \frac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \tag{32}$$

The discriminant of $\mathcal{P}_1(\lambda)$ is equal to:

$$\Delta_1 = \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} + \frac{2}{\sqrt{\beta}}\right)\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} - \frac{2}{\sqrt{\beta}}\right)$$

Using (5)-(6) we get

$$\begin{aligned}
\Delta_1 &= \left(\frac{1}{u_{cd}}\right)^2\left(u_{cc} + \frac{2u_{cd}}{\sqrt{\beta}} + \frac{u_{dd}}{\beta}\right)\left(u_{cc} - \frac{2u_{cd}}{\sqrt{\beta}} + \frac{u_{dd}}{\beta}\right) \\
&= \left(\frac{1}{u_{cd}}\right)^2 \begin{pmatrix} 1 & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} u_{cc} & u_{cd} \\ u_{dc} & u_{dd} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{\beta}} \end{pmatrix} \\
&\times \begin{pmatrix} 1 & -\frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} u_{cc} & u_{cd} \\ u_{dc} & u_{dd} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{\beta}} \end{pmatrix}
\end{aligned}$$

Under the concavity property in Assumption 2, the Hessian matrix of the utility function $u(c, d)$ is quasi-negative definite which implies $\Delta_1 \geq 0$ and the associated characteristic roots are necessarily real. From $\mathcal{P}_2(\lambda)$ we obviously conclude that for any sign of the capital intensity difference b the associated characteristic roots are also necessarily real.

□

7.4 Proof of Proposition 3

Under Assumptions 1-4, let $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$. Using the fact that $\frac{\epsilon_{cc}}{\epsilon_{dc}} = \frac{\epsilon_{cd}}{\epsilon_{dd}}$, we derive the following expression

$$\mathcal{P}_1(\lambda) = \left(\lambda - \frac{\epsilon_{cc}}{\epsilon_{dc}}\right)\left(\lambda - \frac{\epsilon_{dc}}{\beta\epsilon_{cc}}\right) \tag{33}$$

The associated characteristic roots λ_1 and λ_2 are therefore both positive. Moreover we get:

$$\begin{aligned}
\mathcal{P}_1(0) &= \frac{1}{\beta} \geq 1 \\
\mathcal{P}_1(1) &= -\epsilon_{cc}\epsilon_{dc}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}}\right)\left(\frac{1}{\beta\epsilon_{cc}} - \frac{1}{\epsilon_{dc}}\right)
\end{aligned}$$

The normality Assumption 3 implies $\mathcal{P}_1(1) < 0$ and we conclude that the associated characteristic roots λ_1 and λ_2 are such that $\lambda_1 < 1$ and $\lambda_2 > 1$.

From $\mathcal{P}_2(\lambda)$, the associated characteristic roots λ_1 and λ_2 are both positive. Moreover we derive:

$$\mathcal{P}_2(0) = \frac{1}{\beta} \geq 1, \quad \mathcal{P}_2(1) = -\frac{(\beta-b)(1-b)}{\beta b}$$

From constant returns to scale, we get $wa_{01} + ra_{11} = p$ with $a_{01} = l_1/y$ and $a_{11} = k_1/y$. The second equation in (11) rewrites as $p = \beta r$. We then obtain after substitution in the previous equation $r(\beta - a_{11}) = wa_{01} > 0$ and thus

$$\beta - b = \frac{a_{00}(\beta-a_{11})+a_{10}a_{01}}{a_{00}} > 0$$

When $b \geq 0$ we then necessarily have $b < \beta \leq 1$. It follows that $\mathcal{P}_2(0) < 0$ and we conclude that the associated characteristic roots λ_1 and λ_2 are such that $\lambda_1 < 1$ and $\lambda_2 > 1$. The steady state is therefore a saddle-point. □

7.5 Proof of Proposition 4

i) Under Assumptions 1-4, let $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$. As shown previously, we derive from $\mathcal{P}_2(\lambda) = 0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_1(\lambda)$ as given by (33), the associated characteristic roots λ_1 and λ_2 are both negative. Moreover, we get:

$$\mathcal{P}_1(-1) \quad = \quad \left(1 + \frac{\epsilon_{cc}}{\epsilon_{dc}}\right)\left(1 + \frac{\epsilon_{dc}}{\beta\epsilon_{cc}}\right) = \frac{(\epsilon_{cc}+\epsilon_{dc})(\beta\epsilon_{cc}+\epsilon_{dc})}{\beta\epsilon_{cc}\epsilon_{dc}}$$

We conclude easily that

$$\mathcal{P}_1(-1) < 0 \quad \Leftrightarrow \quad \epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$$
$$\mathcal{P}_1(-1) > 0 \quad \Leftrightarrow \quad \epsilon_{cc} \in (-\epsilon_{dc}, -\epsilon_{dc}/\beta)$$

It follows that the steady state is a saddle-point with damped oscillations when $\epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$ and there exists a flip bifurcation with persistent period-2 cycles when ϵ_{cc} crosses the bifurcation values $-\epsilon_{dc}$ or $-\epsilon_{dc}/\beta$.

ii) Under Assumptions 1-4, let $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, and $b < 0$. As shown previously, we derive from $\mathcal{P}_1(\lambda) = 0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_2(\lambda)$, the associated characteristic roots λ_1 and λ_2 are both negative. Moreover we get:

$$\mathcal{P}_2(-1) \quad = \quad \frac{(1+b)(b+\beta)}{\beta b}$$

We conclude easily that

$$\mathcal{P}_1(-1) < 0 \quad \Leftrightarrow \quad b \in (-\infty, -1) \cup (-\beta, 0)$$
$$\mathcal{P}_1(-1) > 0 \quad \Leftrightarrow \quad b \in (-1, -\beta)$$

It follows that the steady state is a saddle-point with damped oscillations when $b \in (-\infty, -1) \cup (-\beta, 0)$. Moreover, if there is some $\beta^* \in (0, 1)$ such that

$b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0,1)$ such that, when β crosses $\bar{\beta}$ from above, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

iii) The case where the consumption good is capital intensive, i.e. $b < 0$, and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$, is obviously derived from the two previous cases.

□

7.6 Proof of Corollary 1

Under a linear homogeneous utility function, standard Euler equalities based on the homogeneity of degree 1, namely $u = u_c c + u_d Bd$, $0 = u_{cc}c + u_{cd}Bd$ and $0 = u_{dc}c + u_{dd}Bd$, lead to

$$u_{cd} = -\frac{u_{cc}}{Bd}, \quad u_{dc} = -\frac{u_{dd}Bd}{c} \quad \text{and thus } u_{dd} = u_{cc}\left(\frac{c}{Bd}\right)^2$$

Moreover, we get from the first order condition $u_d B = \beta u_c$ and (16)

$$\frac{c}{Bd} = \frac{\beta\phi}{1-\phi}$$

Substituting all this into (5)-(6) implies

$$\epsilon_{cd} = -\epsilon_{cc}, \quad \epsilon_{dc} = -\epsilon_{cc}\frac{1-\phi}{\phi}, \quad \epsilon_{dd} = \epsilon_{cc}\frac{1-\phi}{\phi}$$

The characteristic polynomial (15) becomes here

$$\mathcal{P}(\lambda) \quad = \quad \left(\lambda + \frac{\phi}{1-\phi}\right)\left(\lambda + \frac{1-\phi}{\beta\phi}\right)\frac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \tag{34}$$

The result follows from Proposition 4.

□

7.7 Proof of Corollary 2

We consider here Cobb-Douglas technologies as given by (17). We follow the same methodology as in Baierl *et al.* [2]. The Lagrangian associated with the optimization program (1) is:

$$\mathcal{L} = k_0^{\alpha_0} l_0^{1-\alpha_0} + w\big(1 - l_0 - l_1\big) + r\big(k - k_0 - k_1\big) + p\left[k_1^{\alpha_1} l_1^{1-\alpha_1} - y\right] \tag{35}$$

The first order conditions are:

$$r \quad = \quad \alpha_0 k_0^{\alpha_0 - 1} l_0^{1-\alpha_0} = p\alpha_1 k_1^{\alpha_1 - 1} l_1^{1-\alpha_1} \tag{36}$$

$$w \quad = \quad (1-\alpha_0) k_0^{\alpha_0} l_0^{-\alpha_0} = p(1-\alpha_1) k_1^{\alpha_1} l_1^{-\alpha_1} \tag{37}$$

Using $k_0 = k - k_1$, $l_0 = 1 - l_1$, and merging the above equations gives:

$$
\begin{aligned}
l_0^* &= \frac{(1-\alpha_0)\alpha_1(k-k_1^*)}{(\alpha_0-\alpha_1)k_1^*+(1-\alpha_0)\alpha_1 k} && (38)\\
l_1^* &= \frac{\alpha_0(1-\alpha_1)k_1^*}{(\alpha_0-\alpha_1)k_1^*+(1-\alpha_0)\alpha_1 k} && (39)\\
K_c^* &= k-k_1^* && (40)\\
k_1^* &= g(k,y)\equiv g && (41)
\end{aligned}
$$

where

$$g(k,y)=\left\{k_1\in[0,k^{\alpha_1}]\ /\ y=\frac{[\alpha_0(1-\alpha_1)]^{1-\alpha_1}k_1}{[(1-\alpha_0)\alpha_1 k+(\alpha_0-\alpha_1)k_1]^{1-\alpha_1}}\right\} \quad (42)$$

From (36), (38) and (40) we obtain:

$$T_k=r^*=\alpha_0\left[\frac{(1-\alpha_0)\alpha_1}{(1-\alpha_0)\alpha_1 k+(\alpha_0-\alpha_1)g}\right]^{1-\alpha_0} \quad (43)$$

and from (36), (39), (41) and (43):

$$T_y=p^*=\frac{\alpha_0[(1-\alpha_0)\alpha_1]^{1-\alpha_0}[\alpha_0(1-\alpha_1)]^{-(1-\alpha_1)}[(1-\alpha_0)\alpha_1 k+(\alpha_0-\alpha_1)g]^{\alpha_0-\alpha_1}}{\alpha_1} \quad (44)$$

By the derivation of g, we have, for any equilibrium path, the identity $(1-\alpha_0)\alpha_1 k+(\alpha_0-\alpha_1)g=\alpha_0(1-\alpha_1)(g/y)^{1/(1-\alpha_1)}$. Substituting this into (43) and (44) gives after simplifications:

$$
\begin{aligned}
T_k(k,y) &= \alpha_0\left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)}\right)^{1-\alpha_0}\left(\frac{y}{g}\right)^{\frac{1-\alpha_0}{1-\alpha_1}}\\
T_y(k,y) &= -\frac{\alpha_1}{\beta_1}\left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)}\right)^{1-\alpha_0}\left(\frac{y}{g}\right)^{\frac{\alpha_1-\alpha_0}{1-\alpha_1}}\\
T_{kk}(k,y) &= -T_k(k,y)\frac{g_1}{g}
\end{aligned}
$$

with $g_1=\partial g(k,y)/\partial k$. A steady state k^* is then defined as $T_k(k^*,k^*)+\beta T_y(k^*,k^*)$. Denote $g^*=g(k^*,k^*)$ and $y^*=k^*$. Using the derivatives of T in the definition of k^* gives:

$$g^*=\beta\alpha_1 k^* \quad (45)$$

Substituting (45) into the definition of g, we find

$$k^*=\frac{\alpha_0(1-\alpha_1)(\beta\alpha_1)^{\frac{1}{1-\alpha_1}}}{\alpha_1[1-\alpha_0+\beta(\alpha_0-\alpha_1)]} \quad (46)$$

Considering (42), we easily derive

$$g_1=\frac{\beta\alpha_1(1-\alpha_0)(1-\alpha_1)}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \quad (47)$$

From all these results and (4), we get

$$\begin{aligned} c^* = T(k^*, k^*) &= \left(\frac{\alpha_0(1-\alpha_1)}{(1-\alpha_0)\alpha_1}\right)^{\alpha_0} \frac{(1-\alpha_0)(1-\beta\alpha_1)(\beta\alpha_1)^{\frac{\alpha_0}{1-\alpha_1}}}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \\ r^* = T_k(k^*, k^*) &= \alpha_0 \left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)}\right)^{1-\alpha_0} (\beta\alpha_1)^{-\frac{1-\alpha_0}{1-\alpha_1}} \\ T_{kk}(k^*, k^*) &= -\frac{T_k(k^*,k^*)}{k^*} \frac{(1-\alpha_0)^2}{1-\alpha_0+\beta\alpha_1(\alpha_0-\alpha_1)} \\ b &= \frac{\beta(\alpha_1-\alpha_0)}{1-\alpha_0} \end{aligned}$$

We then easily derive

$$\varepsilon_{ck} = \frac{\alpha_0}{1-\beta\alpha_1} \text{ and } \varepsilon_{rk} = \frac{(1-\alpha_0)^2}{1-\alpha_0+\beta\alpha_1(\alpha_0-\alpha_1)}$$

Consider now the characteristic polynomial (48). The characteristic roots are

$$\lambda_1 = -\frac{\phi}{1-\phi}, \ \lambda_2 = -\frac{1-\phi}{\beta\phi}, \ \lambda_3 = \frac{1}{b} \text{ and } \lambda_4 = \frac{b}{\beta} \tag{48}$$

The critical values $\underline{\phi}$ and $\bar{\phi}$ are given in Corollary 1. Assume that $\alpha_0 > (1+\alpha_1)/2$. We immediately derive from the expression of b that $\lambda_3 > -1$ if and only if $\beta > (1-\alpha_0)/(\alpha_0-\alpha_1) \equiv \underline{\beta}$ while $\lambda_4 < -1$. The result follows from the fact that if $\beta = \underline{\beta}$ and $\phi = \bar{\phi}$ or $\underline{\phi}$ then two characteristic roots are simultaneously equal to -1.

□

7.8 Proof of Proposition 5

The characteristic polynomial (15) can be expressed as follows

$$\left[\lambda^2 - \lambda\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{1}{\beta}\right] \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} = -\lambda(\lambda-1)\left(\lambda - \frac{1}{\beta}\right) \frac{\beta}{b\epsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}\right)$$

or equivalently, using the notations of Lemma 1,

$$P_1(\lambda)P_2(\lambda) = P_3(\lambda)$$

with $P_3(\lambda)$ a degree-3 polynomial while $P_1(\lambda)P_2(\lambda)$ is a degree-4 polynomial. If these two polynomials intersect four times, then the four characteristic roots are real. To determine the number of intersections of these polynomials, we can use informations derived from the location of their respective roots. The roots of $P_3(\lambda) = 0$ are quite obvious, namely $\lambda_{31} = 0$, $\lambda_{32} = 1$ and $\lambda_{33} = 1/\beta$. Moreover, depending of the sign of $\epsilon_{cd}, \epsilon_{dc}$ we get

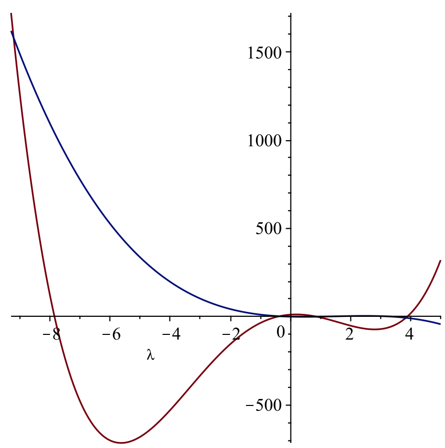
- if $\epsilon_{cd}, \epsilon_{dc} < 0$, then $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} > 0$ and $\lim_{\lambda\to+\infty} P_3(\lambda) = -\infty$ while $\lim_{\lambda\to-\infty} P_3(\lambda) = +\infty$;

- if $\epsilon_{cd}, \epsilon_{dc} > 0$, then $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} < 0$ and $\lim_{\lambda\to+\infty} P_3(\lambda) = +\infty$ while $\lim_{\lambda\to-\infty} P_3(\lambda) = -\infty$;

The roots of $P_1(\lambda)P_2(\lambda) = 0$ are obviously given by the respective roots of $P_1(\lambda) = 0$ and $P_2(\lambda) = 0$.

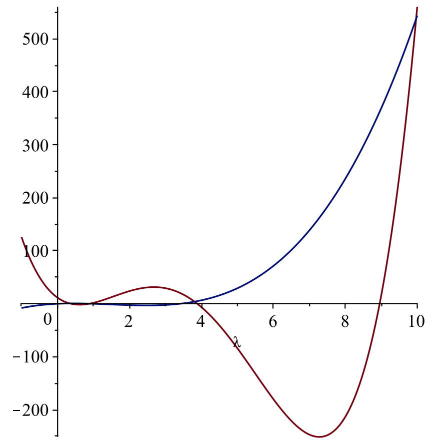
i) Assume first that $b > 0$. We have shown in the proof of Proposition 3 that $b < \beta \leq 1$. The roots of $P_2(\lambda) = 0$ are then quite obvious, namely $\lambda_{21} = 1/b > 1$ and $\lambda_{22} = b/\beta < 1$. Finally, the roots of $P_1(\lambda) = 0$ are necessarily real and negative if $\epsilon_{cd}, \epsilon_{dc} < 0$, or positive if $\epsilon_{cd}, \epsilon_{dc} > 0$. Moreover, we have $\lim_{\lambda \to \pm\infty} P_1(\lambda)P_2(\lambda) = +\infty$ and $P_1(0)P_2(0) > 0$.

If $\epsilon_{cd}, \epsilon_{dc} < 0$, we derive from the above informations that $P_1(b/\beta)P_2(b/\beta) = 0 > P_3(b/\beta)$ while $P_1(1)P_2(1 < P_3(b/\beta) = 0$ implying a first intersection between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Moreover, since $P_1(1/\beta)P_2(1/\beta) < P_3(1/\beta) = 0$ while $P_1(1/b)P_2(1/b) = 0 > P_3(b/\beta)$, we get a second intersection $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Since $P_1(0)P_2(0) > 0$, $P_1(\lambda)P_2(\lambda) = 0$ admits two roots in the negative horthant, $P_3(0) = 0$ and $P_3(\lambda)$ is an increasing function in the negative hortant, we conclude that there necessarily exists a third intersection between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. The last intersection, which also occurs in the negative orthant, is obtained because $\lim_{\lambda \to -\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda \to -\infty} P_3(\lambda)$. Indeed $P_3(\lambda)$ a degree-3 polynomial while $P_1(\lambda)P_2(\lambda)$ is a degree-4 polynomial. We then get the following graphical illustration



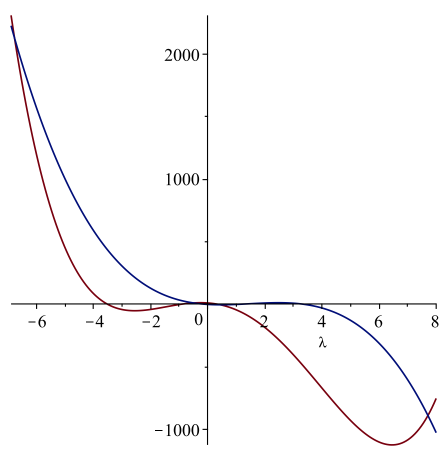
It follows that the four roots of the characteristic polynomial (15) are real.

If $\epsilon_{cd}, \epsilon_{dc} > 0$, the roots of $P_3(\lambda) = 0$ and $P_2(\lambda) = 0$ are the same as before while the roots of $P_1(\lambda) = 0$ are now real and positive. Since $P_1(0)P_2(0) > 0$, $P_1(1/b)P_2(1/b) = 0$ and $P_1(1)P_2(1) > 0$, there necessarily exists a second root of $P_1(\lambda)P_2(\lambda) = 0$ between 0 and $1/b$ implying two intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$. The two others are obtained since $P_1(1/\beta)P_2(1/\beta) > P_3(1/\beta) = 0$, $P_1(b/\beta)P_2(b/\beta) = 0 < P_3(b/\beta)$ and $\lim_{\lambda \to +\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda \to +\infty} P_3(\lambda)$. We then get the following graphical illustration



Here again, it follows that the four roots of the characteristic polynomial (15) are real.

ii) Assume now that $b < 0$ and $\epsilon_{cd}, \epsilon_{dc} > 0$. The roots of $P_2(\lambda) = 0$ become negative, namely $\lambda_{21} = 1/b < \lambda_{22} = b/\beta < 0$. We easily get $P_1(0)P_2(0) > 0$, $P_1(1)P_2(1) < P_3(1) = 0$,$P_1(1/\beta)P_2(1/\beta) < P_3(1/\beta) = 0$, $\lim_{\lambda \to +\infty} P_1(\lambda)P_2(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} P_3(\lambda) = -\infty$. It follows that there are three intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Moreover, we have $\lim_{\lambda \to -\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda \to -\infty} P_3(\lambda)$ implying the existence of two additional intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the negative orthant. We then get the following graphical illustration



and it follows that the four roots of the characteristic polynomial (15) are real.

□

7.9 Proof of Proposition 6

Using a homogeneous of degree $\gamma < 1$ utility function, the degree-4 characteristic polynomial as given by Proposition 2 becomes

$$\begin{aligned} \mathcal{P}(\lambda) &= \left[\lambda^2 + \lambda\left(\tfrac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) + \tfrac{1}{\beta}\right] \tfrac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ &+ \lambda(\lambda-1)\left(\lambda - \tfrac{1}{\beta}\right) \tfrac{\beta}{b}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}\tfrac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \end{aligned} \tag{49}$$

and can be expressed as $\mathcal{Q}_1(\lambda) = \mathcal{Q}_2(\lambda)$ with

$$\begin{aligned} \mathcal{Q}_1(\lambda) &\equiv \tfrac{1}{\gamma-\phi}\left[\lambda^2(\gamma-\phi) + \lambda\left(\tfrac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi[1-\epsilon_{cc}(1-\gamma]}\right) + \tfrac{(\gamma-\phi)}{\beta}\right] \tfrac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) &\equiv -\tfrac{1}{\gamma-\phi}\lambda(\lambda-1)\left(\lambda-\tfrac{1}{\beta}\right)\tfrac{\beta}{b}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}\tfrac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{[1-\epsilon_{cc}(1-\gamma)]} \end{aligned}$$

Considering the limit $\phi \to \gamma$ we immediately conclude that one root λ_1 is necessarily real and equal to $\pm\infty$ and we get

$$\begin{aligned} \mathcal{Q}_1(\lambda) &= \lambda\gamma\tfrac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) &= -\lambda\gamma(\lambda-1)\left(\lambda-\tfrac{1}{\beta}\right)\tfrac{\beta}{b}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}(1-\gamma) \end{aligned}$$

It follows that a second root λ_2 is real and equal to 0. Computing now the derivatives $\mathcal{Q}_1'(\lambda)$ and $\mathcal{Q}_2'(\lambda)$, and evaluating them at 0 gives

$$\begin{aligned} \mathcal{Q}_1'(0) &= \tfrac{\gamma}{\beta} \\ \mathcal{Q}_2'(0) &= -\tfrac{\gamma}{b}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}(1-\gamma) \end{aligned}$$

It follows that $\mathcal{Q}_1'(0) \gtrless \mathcal{Q}_2'(0)$ if and only if $\epsilon_{cc} \lesseqgtr \hat{\epsilon}_{cc}$ with

$$\hat{\epsilon}_{cc} \equiv -\tfrac{b}{(1-\gamma)}\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} \in (0, \tilde{\epsilon}_{cc})$$

Note that $\hat{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$ if and only if

$$-\tfrac{\varepsilon_{ck}}{b\varepsilon_{rk}} > 1 \tag{50}$$

We conclude therefore that under condition (50) there exist two additional intersections between $\mathcal{Q}_1(\lambda)$ and $\mathcal{Q}_2(\lambda)$ implying that the two last characteristic roots λ_3, λ_4 are also real. Let us then assume that $b \in (-\infty, -1) \cup (-\beta, 0)$. We derive that

i) if $\epsilon_{cc} < \hat{\epsilon}_{cc}$ then $\mathcal{Q}_1'(0) > \mathcal{Q}_2'(0)$ with $\mathcal{Q}_1(1/b) = \mathcal{Q}_1(b/\beta) = 0$ which implies that one intersection must occur between -1 and 0, say $\lambda_3 \in (-1, 0)$. Moreover we derive also that $\lambda_1 = -\infty$ and $\lambda_4 < -1$;

ii) if $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$ then $\mathcal{Q}_1'(0) < \mathcal{Q}_2'(0)$ with $\mathcal{Q}_2(1) = 0$ which implies that one intersection must occur between 0 and 1, say $\lambda_3 \in (0, 1)$. Moreover we derive $\lambda_1 = +\infty$ and $\lambda_4 > 1$.

We then conclude by continuity that there exists $0 < \bar{\phi} < \gamma$ such that when $\phi \in (\bar{\phi}, \gamma)$, the above results hold with $\lambda_1 \in (-\infty, -1)$ and $\lambda_2 \in (-1, 0)$ when $\epsilon_{cc} < \hat{\epsilon}_{cc}$ or $\lambda_1 \in (1, \infty)$ and $\lambda_2 \in (0, 1)$ when $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$.

Note now that the characteristic polynomial (49) can be also expressed as $\mathcal{Q}_1(\lambda) = \mathcal{Q}_2(\lambda)$ with

$$\begin{array}{rcl}\mathcal{Q}_1(\lambda) & \equiv & \frac{1}{\phi}\left[\lambda^2\phi + \lambda\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta(\gamma-\phi)\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) + \frac{\phi}{\beta}\right]\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) & \equiv & -\frac{1}{\phi}\lambda(\lambda-1)\left(\lambda - \frac{1}{\beta}\right)\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\phi(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]}\end{array}$$

Considering the limit $\phi \to 0$ we immediately conclude that one root λ_1 is necessarily real and equal to $-\infty$ as $b < 0$, and we get

$$\begin{array}{rcl}\mathcal{Q}_1(\lambda) & = & \frac{\lambda\gamma^2}{\beta[1-\epsilon_{cc}(1-\gamma)]}\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) & = & 0\end{array}$$

It follows that $\lambda_2 = 0$, $\lambda_3 = 1/b$ and $\lambda_4 = b/\beta$ with one larger than -1 and the other lower than -1 as $b \in (-\infty, -1) \cup (-\beta, 0)$. We then conclude by continuity that there exists $0 < \underline{\phi} \leq \bar{\phi}$ such that when $\phi \in (0, \underline{\phi})$, the above results hold with $\lambda_1 \in (-\infty, -1)$ and $\lambda_2 \in (-1, 0)$.

□

7.10 Proof of Proposition 7

The expressions in (29) become here

$$\begin{array}{rcl}B & = & -\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} + \frac{\beta+b^2}{\beta b} \\ & - & \left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) \\ C & = & -\frac{(1+\beta)}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \\ & - & \frac{\beta+b^2}{\beta b}\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) + \frac{2}{\beta}\end{array} \tag{51}$$

As $\epsilon_{cc} < \tilde{\epsilon}_{cc}$ and $b \in (-\infty, -1) \cup (-\beta, 0)$, we immediately get $C > 0$ for any $\phi \in (0, \gamma)$. Moreover, when $\epsilon_{cc} = 0$, we get

$$B = \frac{\beta+b^2}{\beta b} - \frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\gamma(1-\gamma)}{\gamma-\phi} - \left(\frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)}\right) < 0$$

for any $\phi \in (0,\gamma)$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{\gamma-\phi}{\beta\gamma(1-\gamma)}\left[\frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)} - \frac{\beta+b^2}{\beta b}\right] \tag{52}$$

As the right-hand-side of (52) is a decreasing function of ϕ, we conclude that it is always satisfied if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{1}{\beta(1-\gamma)} \equiv \varepsilon^1 \tag{53}$$

with $\varepsilon^1 > 1$. Therefore, under condition (53) there exists $\bar{\epsilon}^1_{cc} \in (0, \tilde{\epsilon}_{cc})$ such that $B < 0$ for any $\phi \in (0,\gamma)$ if $\epsilon_{cc} \in (0, \bar{\epsilon}^1_{cc})$.

Let us consider now the expression $P = 8C - 3B^2$. We derive from (51) that P is a hump-shaped function of ϕ over $(0,\gamma)$. When $\epsilon_{cc} = 0$, we get

$$\begin{aligned} C &= -\frac{1+\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\gamma(1-\gamma)}{\gamma-\phi} - \frac{\beta+b^2}{\beta b}\left(\frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)}\right) + \frac{2}{\beta} \\ &\equiv -\frac{1+\beta}{b}x - \frac{\beta+b^2}{\beta b}z + \frac{2}{\beta} \\ B &= \frac{\beta+b^2}{\beta b} - z - \frac{\beta}{b}x \end{aligned} \tag{54}$$

and

$$P < -\frac{8(1+\beta}{b}x - \left(\frac{\beta+b^2}{\beta b} + z\right)^2 - 2\left[\left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{8}{\beta} + z^2\right]$$

Straightforward computations yield $z \geq 2/\sqrt{\beta}$ and thus

$$\begin{aligned} \left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{8}{\beta} + z^2 > \left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{4}{\beta} &= \left(\frac{\beta+b^2}{\beta b} - \frac{2}{\sqrt{\beta}}\right)\left(\frac{\beta+b^2}{\beta b} + \frac{2}{\sqrt{\beta}}\right) \\ &= \frac{(b-\sqrt{\beta})^2}{\beta b}\frac{(b+\sqrt{\beta})^2}{\beta b} > 0 \end{aligned}$$

for any $\phi \in (0,\gamma)$. Therefore, $P < 0$ for any $\phi \in (0,\gamma)$ when $\epsilon_{cc} = 0$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{\gamma-\phi}{8(1+\beta)\gamma(1-\gamma)}\left\{\left(\frac{\beta+b^2}{\beta b} + z\right)^2 + 2\left[\left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{8}{\beta} + z^2\right]\right\} \tag{55}$$

We can show that the right-hand-side of (55) is a U-shaped function of ϕ over $(0,\gamma)$ and there exists a unique minimum value $\varepsilon^2 > 1$ such that condition (55) holds if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \varepsilon^2 \tag{56}$$

It follows that under condition (56) there exists $\bar{\epsilon}^2_{cc} \in (1, \bar{\epsilon}^1_{cc})$ such that $P < 0$ for any $\phi \in (0,\gamma)$ if $\epsilon_{cc} \in (0, \bar{\epsilon}^2_{cc})$.

Let us consider finally R and D as given by (30) and (31). Straightforward computations yield:

$$\lim_{\phi\to 0} B = -\infty \text{ and } \lim_{\phi\to 0} C = -\infty \text{ so that } \lim_{\phi\to 0} R = -\infty \text{ and } \lim_{\phi\to 0} D = -\infty$$

and there exists $\underline{\gamma}^1 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^1, 1)$

$$\lim_{\phi \to \gamma} B = -\infty \text{ and } \lim_{\phi \to \gamma} C = -\infty \text{ so that } \lim_{\phi \to \gamma} R = -\infty \text{ and } \lim_{\phi \to \gamma} D = -\infty$$

We need now to show that there exists a subset of values of ϕ for which R and D can be positive. Let us consider the particular values $\epsilon_{cc} = 0$, and $b = -\beta$. It follows from (54) that

$$B^2 + \tfrac{8}{\beta} - 4C = \left(z(\phi) - x - \tfrac{1+\beta}{\beta}\right)^2 - \tfrac{8(1+\beta)x}{\beta} \equiv F(\phi)$$

with

$$z(\phi) = \tfrac{(\gamma-\phi)^2 + \beta\phi^2}{\beta\phi(\gamma-\phi)} \text{ and } x = \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \tfrac{\gamma(1-\gamma)}{\gamma-\phi}$$

Obviously, $F(\phi) = 0$ can be solved through the degree two polynomial

$$z(\phi) - x - \tfrac{1+\beta}{\beta} = 2\sqrt{\tfrac{2(1+\beta)x}{\beta}}$$

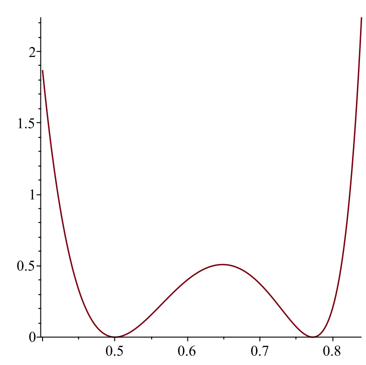
It follows therefore that there exists $\underline{\gamma}^2 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^2, 1)$ the two roots for which $F(\phi) = 0$ satisfy $\phi_1, \phi_2 \in (0,1)$. In the particular case $\gamma = 1$, these roots are indeed such that

$$\phi_1 = \tfrac{1}{2} \text{ and } \phi_2 = \tfrac{1}{1+\beta}$$

Moreover, there exists $\underline{\gamma}^3 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^3, 1)$ there is a value $\phi_3 \in (\phi_1, \phi_2)$ such that $F'(z) = 0$ when $\phi = \phi_1, \phi_2, \phi_3$. Notice indeed that in the particular case $\gamma = 1$, we have

$$\phi_3 = \tfrac{1}{1+\sqrt{\beta}}$$

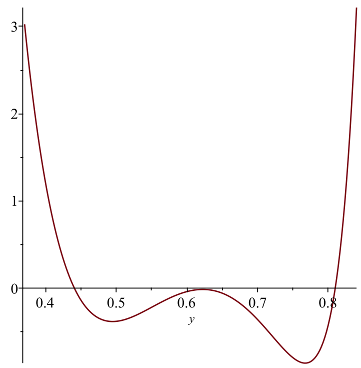
Obviously, $F(\phi) > 0$ when $\phi = \phi_3$. Note also that $\lim_{\phi \to 0} F(\phi) = \lim_{\phi \to 1} F(\phi) = +\infty$. As a result, we conclude that $F(\phi) \geq 0$ for any $\phi \in (0,1)$ with the following shape



Consider now $\gamma < 1$, $\epsilon_{cc} > 0$ and the expressions of B and C as given by (51), and let us define

$$B^2 + \frac{8}{\beta} - 4C \equiv G(\epsilon_{cc}, \gamma, \phi) \tag{57}$$

By continuity, there exists $\underline{\gamma}^4 \in (0, 1)$ close to 1 and $\tilde{\phi}_3$ close to ϕ_3 such that for any given $\gamma \in (\underline{\gamma}^4, 1)$, $\partial G(\epsilon_{cc}, \gamma, \tilde{\phi}_3)/\partial\phi = 0$. Moreover, since $G(\epsilon_{cc}, \gamma, \phi)$ is a decreasing function of ϵ_{cc} with $\lim_{\epsilon_{cc} \to \bar{\epsilon}_{cc}^2} G(\epsilon_{cc}, \gamma, \tilde{\phi}_3) < 0$, we conclude that there exists $\underline{\epsilon}_{cc} \in (0, \bar{\epsilon}_{cc}^2)$ such that for any given $\gamma \in (\underline{\gamma}^4, 1)$, when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we have $G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3) = \partial G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3)/\partial\phi = 0$ such that



We conclude therefore that there exist $\bar{b} \in (-\beta, 0)$, $\underline{\phi}^c \in (0, \phi_1)$ and $\bar{\phi}^c \in (\phi_2, \gamma)$ such that if $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4\}, 1)$, $b \in (-\beta, \bar{b})$ and $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$, then $R > 0$ when $\phi \in (\underline{\phi}^c, \bar{\phi}^c)$ and $R < 0$ when $\phi \in (0, \underline{\phi}^c) \cup (\bar{\phi}^c, \gamma)$.

Let us consider now D. We have proved that for any given $\gamma \in (\underline{\gamma}^4, 1)$, if $\epsilon_{cc} \in (0, \bar{\epsilon}_{cc}^2)$ then $P < 0$ for any $\phi \in (0, \gamma)$. This implies that $-3B^2 < -8C$ and thus

$$\begin{aligned} \frac{8}{\beta} - 3B^2 + 4C \quad < \quad & \frac{8}{\beta} - 4C = -4\Bigg\{ - \frac{(1+\beta)}{b} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{(1-\gamma)[\gamma - \epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \\ & - \quad \frac{\beta + b^2}{\beta b} \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]} \right) \Bigg\} < 0 \end{aligned}$$

It follows that if $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4\}, 1)$, $b \in (-\beta, \bar{b})$ and $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$, then D has the same sign as R for any $\phi \in (0, \gamma)$, and the characteristic roots are complex when $\phi \in (\underline{\phi}^c, \bar{\phi}^c)$ and real when $\phi \in (0, \underline{\phi}^c) \cup (\bar{\phi}^c, \gamma)$. Moreover, when $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$, $R = D = 0$.

As explained in Remark 1, the polynomial (28) belongs to the class of quasi-palindromic equation and the exact solutions can be computed. Dividing $\mathcal{P}(\lambda)$ by λ^2 gives

$$\mathcal{P}(\lambda) \quad = \quad \lambda^2 + \left(\frac{1}{\lambda\beta}\right)^2 - B\left(\lambda + \frac{1}{\lambda\beta}\right) + C = 0$$

and denoting $z = \lambda + 1/(\lambda\beta)$ yields to the following degree-2 polynomial in z

$$\mathcal{P}(z) \quad = \quad z^2 - zB + C - \frac{2}{\beta}$$

The corresponding discriminant is then

$$\Delta_z = B^2 + \frac{8}{\beta} - 4C = \frac{R}{B}$$

and under the previous conditions we have $\Delta_z < 0$. The roots are then

$$z_1 = \frac{B + i\sqrt{-\frac{R}{B}}}{2} \text{ and } z_2 = \frac{B - i\sqrt{-\frac{R}{B}}}{2}$$

Plugging this into the definition of z gives the following two degree-2 polynomials in λ:

$$\lambda^2\beta - \lambda z_1\beta + 1 = 0 \text{ and } \lambda^2\beta - \lambda z_2\beta + 1 = 0$$

Denoting $\Delta_1 = (z_1\beta)^2 - 4\beta$ and $\Delta_2 = (z_2\beta)^2 - 4\beta$, straightforward computations give

$$\sqrt{\Delta_1} = \frac{\beta\left(\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}} + i\frac{B\sqrt{\frac{-R}{B}}}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right)}{2}$$

$$\sqrt{\Delta_2} = \frac{\beta\left(\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}} - i\frac{B\sqrt{\frac{-R}{B}}}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right)}{2}$$

and we finally derive the characteristic roots

$$\lambda_1 = \frac{B+\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}} + i\sqrt{\frac{-R}{B}}\left[1+\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

$$\lambda_2 = \frac{B+\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}} - i\sqrt{\frac{-R}{B}}\left[1+\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

$$\lambda_3 = \frac{B-\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}} + i\sqrt{\frac{-R}{B}}\left[1-\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

$$\lambda_4 = \frac{B-\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}} - i\sqrt{\frac{-R}{B}}\left[1-\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

with $\lambda_3 = 1/(\beta\lambda_1)$ and $\lambda_4 = 1/(\beta\lambda_2)$. The existence of a Hopf bifurcation amounts to show that the product $\lambda_1\lambda_2$ can cross the value 1 when the parameter ϕ is varied over the interval $(\underline{\phi}^c, \bar{\phi}^c)$. Obviously we get

$$\lambda_1\lambda_2 = \left(\frac{B+\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}{4}\right)^2 \frac{B^2-\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}$$

By definition we know that if $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$, we get $R = 0$ and thus

$$\lambda_1\lambda_2 = \left(\frac{B+\sqrt{B^2-\frac{16}{\beta}}}{4}\right)^2$$

Considering that $B < 0$, we then derive that $\lambda_1\lambda_2 < 1$ if and only if

$$B < -\tfrac{2(1+\beta)}{\beta} \tag{58}$$

But since $R = 0$, $B^2 = 4C - 8/\beta$ and, using (51) and assuming $b = -\beta$, inequality (61) becomes

$$\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \tfrac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} + \tfrac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]} > \tfrac{1+\beta}{\beta} \tag{59}$$

When $\epsilon_{cc} = 0$, this inequality becomes

$$\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \tfrac{\gamma(1-\gamma)}{(\gamma-\phi)} + \tfrac{\gamma-2\phi}{\phi(1-\phi)} \tfrac{\gamma-\phi(1+\beta)}{\beta} > 0 \tag{60}$$

There exists $\underline{\gamma}^5 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^5, 1)$, (60) is obviously satisfied when $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$. Since the left-hand-side of inequality (59) is an increasing function of ϵ_{cc}, we conclude that $\lambda_1\lambda_2 < 1$ when $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5\}, 1)$, $b \in (-\beta, \bar{b})$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$ and $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$.

Tedious but straightforward computations also show that $\lambda_1\lambda_2$ is a hump-shaped function of ϕ over $(\underline{\phi}^c, \bar{\phi}^c)$. Consider the critical values $\underline{\epsilon}_c$ and $\tilde{\phi}_3$ previously defined such that when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we have $G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3) = \partial G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3)/\partial\phi = 0$ with $G(.)$ as defined by (57). We know that $\tilde{\phi}_3$ is in a neighborhood of $\phi_3 = 1/(1 + \sqrt{\beta})$. It follows that when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we get again $R = 0$ and following the same argument as above we conclude that $\lambda_1\lambda_2 > 1$ if and only if

$$B > -\tfrac{2(1+\beta)}{\beta} \tag{61}$$

Assuming $b = -\beta$ and $\phi = \phi_3$, this inequality is approximated by

$$\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \tfrac{(1-\gamma)[1+\sqrt{\beta}-\underline{\epsilon}_{cc}\phi(1-\gamma)]}{1-\underline{\epsilon}_{cc}(1-\gamma)} + \tfrac{2-\underline{\epsilon}_{cc}(1-\gamma)(1-\sqrt{\beta})}{1-\underline{\epsilon}_{cc}(1-\gamma)} < \tfrac{1+\beta}{\sqrt{\beta}} \tag{62}$$

When $\gamma = 1$, this inequality is obviously satisfied. Therefore, there exists $\underline{\gamma}^6 < 1$ such that $\lambda_1\lambda_2 > 1$ when $\gamma \in (\underline{\gamma}^6, 1)$, $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$. We conclude that there exists $\bar{\epsilon}_{cc}^3 \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2]$, $\underline{\phi}^H \in (\underline{\phi}^c, \tilde{\phi}_3)$ and $\bar{\phi}^H \in (\tilde{\phi}_3, \bar{\phi}^c)$ such that when $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5, \underline{\gamma}^6\}, 1)$, $b \in (-\beta, \bar{b})$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^3)$ then $\lambda_1\lambda_2 < 1$ when $\phi \in (\underline{\phi}^c, \underline{\phi}^H) \cup (\bar{\phi}^H, \bar{\phi}^c)$ and $\lambda_1\lambda_2 > 1$ when $\phi \in (\underline{\phi}^H, \bar{\phi}^H)$. The result follows denoting $\underline{\gamma} = \max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5, \underline{\gamma}^6\}$, $\bar{\varepsilon} = \min\{\varepsilon^1, \varepsilon^2\}$ and $\bar{\epsilon}_{cc} = \min\{\bar{\epsilon}_{cc}^1, \bar{\epsilon}_{cc}^2, \bar{\epsilon}_{cc}^3\}$.

□

7.11 Proof of Proposition 8

As shown in the proof of Proposition 1, there exists a unique steady state (k^*, d^*) solution of equations $R^* = r^*/p^* = \beta^{-1}$ and $u_d(c^*, Bd^*) = \beta u_c(c^*, Bd^*)$. Moreover, k^* does not depend on the utility function $u(c, Bd)$. Since the stationary bequest x^* is strictly positive if and only if $r^*k^* = T_k(k^*, k^*)k^* > d^*$, let us consider a particular value $d^* = \bar{d} \in (0, \min\{T_k(k^*, k^*), T_k(k^*, k^*)k^*\})$. Then, for any $\beta \in (0, 1)$, the same argument as in the proof of Proposition 1 holds: there generically exists a unique value B^* such that when $B = B^*$, $d^* = \bar{d}$ is a normalized steady state such that $x^* > 0$.

□

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