Optimal Quantitative Easing in a Currency Union*

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- -Preliminary. Please do not circulate- -

Abstract

In practice, quantitative easing (QE) in a currency union, such as the one implemented by the European Central Bank (ECB) in the Euro Area, follows guidelines arising from the *capital key*, which allocates country-specific long-term government bond purchases by GDP and population weights. This paper analyzes the optimal allocation of long-term government bond purchases within a currency union using a two-region DSGE model where regions are asymmetric with respect to portfolio characteristics. QE affects government asset prices in three ways: 1) it directly lowers the term premium component of long-term yields, 2) term premiums spill over through portfolio rebalancing of cross-border assets within the union, 3) lower outstanding government debt held by private agents lowers the risk premium on these assets. An optimal quantitative easing policy under discretion does not only reflect different region sizes, but is also a function of parameters dictating imperfect substitutability between assets, the extent of price rigidities, and other dimensions of heterogeneity across regions.

Keywords: Currency union, DSGE model, optimal monetary policy, portfolio rebalancing, quantitative easing, zero-lower bound

JEL: E43, E52, E58, F45

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1 Introduction

In early 2015 the European Central Bank (ECB) joined the group of central banks that have implemented large-scale asset purchase programs as unconventional policy measures. These asset purchases, also called quantitative easing (QE), have led to a strong extension of the central banks' balance sheets. The ECB's largest QE program, announced in January 2015 (Public Sector Purchase Program), foresaw buying €60 billion of assets a month from March 2015 to September 2016, which in sum corresponds to circa 10% of annualized euro area (EA) GDP. In December 2015, the ECB extended the program until March 2017, and it raised the amount of monthly purchases to €80 billion starting from April 2016. In December 2016 the program was extended and modified again, lengthening the period of asset purchases until December 2017, but at a reduced pace of €60 billion of assets a month after March 2017. In October 2017 the program was amended once more extending asset purchases at a reduced monthly pace of €30 billion. In December 2018, the ECB decided to stop asset purchases at the end of 2018, whereas maturing bonds will still be reinvested.

The objective of this paper is twofold. We first provide a quantitative evaluation of the macroeconomic effects of quantitative easing in a currency union, such as the Euro Area, using a two-region dynamic stochastic general equilibrium (DSGE) model consisting of the EA Periphery and Core. As we will show later on, the regions are asymmetric along a number of dimensions, but most importantly for the effects of the QE policy, they also differ in their portfolio characteristics. Our second objective is to investigate the optimality of the allocation of long-term government bond purchases within the currency union. In practice, the ECB quantitative easing policy is designed so that purchases of long-term government bonds follow the *capital key*. That is, purchases from each region are assigned weights based on each country's economic (GDP) and geographic (population) size. However, an optimal allocation of quantitative easing will now not only reflect the different size of each region, but will also be a function of the dimensions of heterogeneity across regions related to portfolio characteristics.

In order to capture the effect of QE policies and break Wallace's irrelevance theorem we introduce a specific financial friction that limits investors' ability to arbitrage assets of different origin and maturity in their portfolio. A "transaction cost" that agents have to pay when adjusting their portfolio giving rise to imperfect substitutability between assets of different classes and isolates the *portfolio rebalancing* channel of QE, as first introduced by Chen *et al.* (2012).

Imperfect substitutability is assumed to take the form as in Alpanda and Kabaca (2019). As in that study, we assume that the asset portfolio is a CES aggregate of sub-portfolios of short-term and long term bonds. The sub-portfolios are in turn nested CES aggregates of domestic and foreign bonds of the same maturity. The approach is suitable for a framework of a currency union as it captures spillovers of quantitative easing that arise due to changes in domestic and foreign term premia. Allowing for cross-border holdings of assets of different maturities has been shown to play a key role in shaping the macroeconomic effects in DSGE-model based analyses of ECB QE in the Euro Area (see Priftis and Vogel (2017); Kolasa and Wesolowski (2018); Hohberger *et al.* (2019)).

In modeling QE through this device, we implicitly assume that investors have preferences for assets of different classes following a 'preferred habitat' motive, similar to Vayanos and Vila (2009), capturing the notion that relative asset prices depend on their relative supply. Due to imperfect substitutability, the central bank can use asset purchases to alter the relative supplies of assets and hence bond returns, i.e. to flatten the yield curve. In our model, central bank long-term asset purchases have effects on the real economy through the extent to which private investors are induced to re-establish the portfolio mix of short-term and long-term assets holdings, issued both domestically and abroad. These purchases affect asset prices and by extension real variables. First, purchases directly lower the term premium component of long-term yields. Second, term premiums spill over within the union through portfolio rebalancing of assets across borders. And third, lower outstanding government debt held by private agents also lowers the risk premium on these assets. Ultimately, reduced savings strengthen contemporaneous consumption demand.

The magnitude of these effects is driven by the extent to which assets are imperfectly substitutable. More precisely, they are driven by the elasticities of substitution across asset classes in the CES (sub)-portfolios. To calibrate these parameters is an arduous task as observed changes in (relative) short and long, or domestic and foreign, bond shares over time are not uniquely driven by the quantitative easing policy of the ECB. We place empirical discipline on these parameters by exploiting the detailed data from the ECB's Security Holdings Statistics. We identify a QE shock using sign restrictions on movements of short-term debt, long-term debt, their domestic and foreign counterparts, as well as interest rate yields, by estimating an SVAR for the Periphery and the Core from 2013Q4 to 2018Q4. We then set values for the elasticities of substitution across assets of different maturity and origin in our quantitative model so that it replicates the same impulse responses as those in the data. In both regions, GDP and inflation increase following a QE shock, but the effect is larger for the Periphery. This is consistent with the larger decline in the 10-year government bond yields for the Periphery. At the same time, total debt increases by more in the Periphery, and long-term debt declines by more in the Core. The results suggest that financial frictions are higher in the Periphery so for a given size of QE shock across regions Periphery portfolios rebalance to a larger degree than in Core.

In our quantitative model a 10% QE shock generates a larger drop in the term premium in the Periphery (75 basis points (bp)) than in the Core (30bp), implying a more expansionary effect on Periphery consumption than in that of the Core.

In our simulations we assume that the union wide central bank allocates its long-term bond purchases across regions by following the guidelines from the ECB's *capital key*. This means that long-term bonds are purchased from each region but weighed according to the relative GDP and population shares. Therefore, a larger share of long-term government bonds are purchased from e.g., Germany, while many less from e.g., Portugal and Greece. Dividing up the Euro area into the most representative economies of the Periphery and Core this translates to 40% of the central bank balance sheet injected into the Periphery (Spain, Portugal, Ireland) and the remaining 60% into the Core (Germany, France, Netherlands).

However, it is clear that countries within the currency union are heterogeneous in additional dimensions. Most relevant for the design of a QE policy are the asymmetries that affect the strength of its transmission channel, which for the purposes of this paper is assumed to operate through portfolio rebalancing. We therefore document a set of asymmetries across countries related to portfolio *characteristics* and comprise of differences in: i) portfolio holdings of long-term debt vis-a-vis short-term debt, ii) home bias in debt portfolios, and iii) the degree of imperfect substitutability across assets of different maturity and origin. All these asymmetries are important in dictating the effects of QE through the term premium and aggregate demand while the latter can be interpreted as a measure of the extent of differing financial frictions between countries.

After having established that the macroeconomic effects rely crucially on the way long-term bond purchases are allocated across regions, the natural question that arises is whether the allocation is optimal? To investigate this we set up an optimal quantitative easing policy under discretion, where the union-wide central bank cannot commit to future policy plans. As in Harrison (2017) we seek a Markov perfect policy in which both the optimal path of quantitative easing, as well as its distribution across regions, is a function only of the relevant state variables in the model. As we show, the optimal share of quantitative easing will now not only reflect the different size of each region, but will also be a function of the dimensions of heterogeneity across regions related to portfolio characteristics.

Related literature on ECB QE

Research at the ECB has provided evidence for the impact of unconventional monetary tools on long-term bond yields and other asset prices through portfolio reallocation. An event study by Altavilla *et al.* (2015) reports a 30-50 basis-point (bp) decline in 10-year government bond yields, lower corporate bond spreads, higher equity prices, and euro depreciation. Andrade *et al.* (2016) report a decline of EA 10-year government bond yields in the range of 27-64 bp, higher equity prices and inflation expectations. De Santis (2016) finds an average decline in 10-year government bond yields by 63 bp between September 2014 and October 2015.

Recently, there has also been a growing literature on the effects of QE operating through the portfolio rebalancing channel in model-based analyses. In their majority, these papers feature imperfect asset substitutability that originates from a transaction cost motive, that is, an asset maturity composition decision subject to adjustment costs.

Hohberger *et al.* (2019) assess the macroeconomic effects of the ECB's QE program using a DSGE model of of the EA estimated with Bayesian techniques and employ a methodological extension that measures the non-linear contribution of QE in shock decompositions under an occasionally ZLB solved in a piecewise linear fashion. Their results suggest an average contribution of ECB QE to annual Euro Area GDP growth and CPI inflation in 2015 – 2018 of 0.4 and 0.9 percentage points, respectively, with a maximum impact in 2016. Using a version of the Gertler and Karadi (2013) model for the EA, Andrade *et al.* (2016) assume an AR(2) specification of the QE shock with a size equal to 11.4% of EA GDP at the peak. They find the ECB asset purchase program (APP) to increase inflation by 40 bp and output by 1.1 percent at their peak, which is reached after around 2 years. In a DSGE model with shadow EONIA rate, Mouabbi and Sahuc (2019) find that EA year-on-year GDP growth and inflation would have been lower by 1.1 pp and 0.6 pp, respectively, on average in 2014-17 in the absence of unconventional monetary policies. Sahuc (2016), using the framework of Gertler and Karadi (2013), finds effects of ECB QE (9% of EA GDP) on EA real GDP growth (inflation) of 0.2 (0.1 pp) in 2015-16 for short-term rates constant in 2015, whereas keeping the policy rate unchanged for another year raises the average growth (inflation) effect in 2015-16 to 0.6 pp (0.6 pp). Cova *et al.* (2015) study the impact of the ECB's asset purchase program (APP) in a multi-country DSGE model with imperfect substitutability between assets of different maturity, motivated by the differing liquidity services they provide. A QE shock corresponding to monthly purchases of 60 billion euros and lasting for 7 quarters (and subsequently being phased out) increases the level of GDP and inflation in the EA by approximately 1 pp over 2015-17.

The remainder of the paper is structured as follows: Section 2 outlines a model for a currency union and describes the channels through which quantitative easing affects the Periphery and Core; Section 3 discusses the calibration of parameters and presents a quantitative analysis of long-term asset purchases by the union central bank; Section 4 outlines an optimal quantitative easing program and optimal allocation of purchases across regions; Section 5 summarizes the paper and concludes. A description of the optimal quantitative easing program is provided in Appendix A.

2 Asymmetries in Portfolio Characteristics

Figures 2.1, 2.2, and 2.3 illustrate a set of asymmetries across regions in the currency union related to portfolio characteristics. These asymmetries are important for the transmission mechanism of QE through the portfolio rebalancing channel in a currency union.

Figure 2.1 shows how the evolution of the short share (defined as the ratio of short-term debt over total debt) has evolved since 2013Q4 for the Periphery and the Core. The short share has diverged over time reaching a maximum difference in 2018Q4. The level of the short share has consistently been lower for the

Periphery (around 0.3% on average, while 0.5% on average for the Core).

Figure 2.1 shows the evolution of the home bias in the short share for the Periphery and the Core over the same period. Home bias in the short share is defined as the ratio of domestic short-term debt over total short-term debt, where total short-term debt is the sum of domestic short-term debt and foreign short-term debt (short-term debt includes holdings of the monetary base). For both regions, home bias in the short share is high, with more than 95% of short-term bonds being held domestically. However, home bias in the short share has been more volatile for the Periphery, but consistently lower than in the Core throughout the entire horizon (around 0.97% on average compare to 0.995% in the Core).

Finally, 2.3 shows the home bias in the long share. The series are roughly constant from 2013Q4 for both regions, but home bias in long-term debt is significantly lower in the Periphery. It averages around 60%, whereas is almost 100% in the Core.

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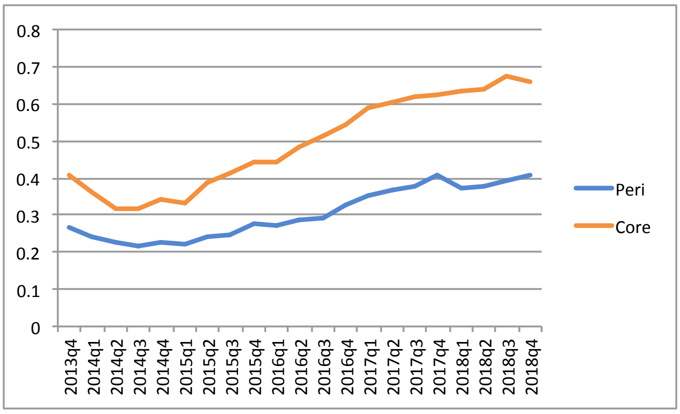
3 A Model for a Currency Union

The union is populated by a continuum of households of measure 1 and consists of two regions, the Core and the Periphery. In each region, there is a continuum of agents, with population size equal to the number of differentiated goods. Agents in the Periphery span the interval $[0, n]$ while agents in the Core the interval $(n, 1]$. In each region, households derive utility from the consumption of goods produced in both regions and supply labor to firms located domestically. Each household has access to all financial markets and can trade in assets of different maturities across borders. Changes in the relative supply of long-term assets produce real effects because short- and long-term assets, as well as home and foreign assets of both maturities, are imperfect substitutes.

Monetary policy is conducted by the union wide central bank controlling the nominal interest rate, which coincides with the rate on short-term assets. The central bank can also engage in unconventional policy through purchases of long-term government bonds. Finally, we assume that assets, regardless of their maturity, are not in zero net supply. Instead, the government in each region accumulates debt, and along with lump-sum transfers, finances expenditures.

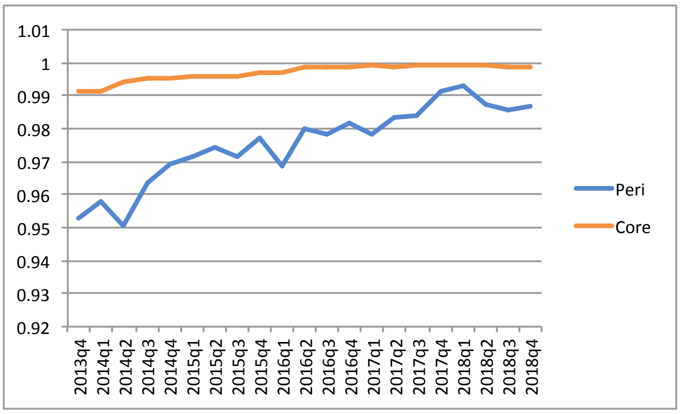
In what follows, we treat the Periphery as the home country, and variables denoted with $*$ refer to the Core. Since the regions are symmetric, we only describe the problem of the Periphery.

Figure 2.1: Short share



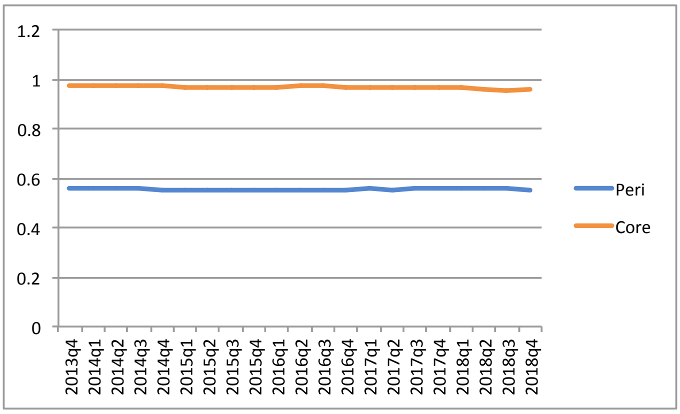
Notes: The share of short-term debt over total debt. Total debt is the sum of short-term debt and long-term debt. Short-term debt includes holdings of the monetary base. Periphery consists of Spain, Italy, Ireland; Core consists of Germany, France, Netherlands.

Figure 2.2: Home bias in short share



Notes: The share of domestic short-term debt over total short-term debt. Total short-term debt is the sum of domestic short-term debt and foreign short-term debt. Short-term debt includes holdings of the monetary base. Periphery consists of Spain, Italy, Ireland; Core consists of Germany, France, Netherlands.

Figure 2.3: Home bias in long share



Notes: The share of domestic long-term debt over total long-term debt. Total long-term debt is the sum of domestic long-term debt and foreign long-term debt. Periphery consists of Spain, Italy, Ireland; Core consists of Germany, France, Netherlands.

3.1 Households

The representative household i in the Periphery derives utility from consumption $C_t(i)$ and disutility from supplying labor $L_t(i)$ to domestic firms. The expected utility function is given by

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(\log C_\tau(i) - \frac{L_\tau(i)^{1+\gamma}}{1+\gamma} \right) \tag{3.1}$$

where β is the subjective discount factor and γ is the inverse of the Frisch elasticity of labor supply. The consumption aggregate $C_t(i)$ is a composite of Periphery, $C_{H,t}$, and Core,$C_{F,t}$, good indices and is given by

$$C_t(i) = \left[\zeta^{\frac{1}{\kappa}} C_{H,t}(i)^{\frac{\kappa-1}{\kappa}} + (1-\zeta)^{\frac{1}{\kappa}} C_{F,t}(i)^{\frac{\kappa-1}{\kappa}} \right]^{\frac{\kappa}{\kappa-1}} \tag{3.2}$$

where κ captures the intratemporal elasticity of substitution between home and foreign goods, and ζ reflects the weights of domestic and foreign consumption goods in the aggregate bundle. ζ is a function of the relative size of the economy, n, and the degree of home bias λ, so $1-\zeta = (1-n)\lambda$. Consumption indices in the Periphery for household i are defined as

$$C_{H,t}(i) = \left[\left(\frac{1}{n}\right)^{1/\theta} \int_0^n c_t(i,h)^{\frac{\theta-1}{\theta}} dh \right]^{\frac{\theta}{\theta-1}}, \quad C_{F,t}(i) = \left[\left(\frac{1}{1-n}\right)^{1/\theta} \int_n^1 c_t(i,f)^{\frac{\theta-1}{\theta}} df \right]^{\frac{\theta}{\theta-1}} \tag{3.3}$$

where $c_t(i,h)$ and $c_t(i,f)$ are consumption of Periphery brand h and Core brand f by the Periphery household i at time t. θ is the elasticity of substitution of goods produced within the same region.

3.1.1 Budget constraint

Household i's budget constraint writes as follows:

$$\begin{gathered} C_t(i) + b_{HS,t}(i) + b_{FS,t}(i) + q_{L,t} b_{HL,t}(i) + q^*_{L,t} b_{FL,t}(i) \\ \leq (1-\tau^w_t) w_t L_t(i) + \frac{R_{t-1} b_{HS,t-1}(i)}{\pi_t} + \frac{\psi_t R^U_{t-1} b_{FS,t-1}(i)}{\pi_t} + \frac{(1+\rho q_{L,t}) b_{HL,t-1}(i)}{\pi_t} \\ + \frac{\psi_t (1+\rho q^*_{L,t}) B_{FL,t-1}(i)}{\pi_t} + \frac{\Xi_t}{\pi_t} - [\Psi - \psi \alpha_t] \end{gathered} \tag{3.4}$$

where w_t denotes the real wage, τ^w_t is a labor income tax, Ξ_t are nominal profits from firm ownership, and π_t is gross inflation. Each household has access to all financial markets and can trade in assets of different maturities: domestic short-term, $b_{HS,t}$, and long-term, $b_{HL,t}$, bonds, as well as foreign short-term, $b_{FS,t}$, and long-term, $b_{FL,t}$, bonds, where $b_{HS,t} = B_{HS,t}/P_t$ and similar expressions hold for long-term bonds. The price of a domestic and foreign short-term bond is R_t and R^U_t, respectively, where R^U_t is the union central bank policy rate. Long-term interest rates are determined as $R_{L,t} = \frac{1+\rho q_{L,t}}{q_{L,t-1}}$ and $R^*_{L,t} = \frac{1+\rho q^*_{L,t}}{q^*_{L,t-1}}$ for the home and foreign bonds, reflecting imperfect substitution between assets of different maturities.

The last component of the budget constraint, Ψ, is a term capturing transaction costs in the asset portfolio, α_t. This serves to generate real effects from changes in the supply of long-term bonds above and beyond those created by imperfect substitutability. These transaction costs can be alleviated by households by carrying liquid assets in their portfolio.

3.1.2 Portfolio composition

Following the approach of Alpanda and Kabaca (2019) we assume that the asset portfolio, α_t, is a CES aggregate of sub-portfolios of short-term, $\alpha_{S,t}$, and long term bonds, $\alpha_{L,t}$, respectively:

$$\alpha_t\left(i\right) = \left[\zeta_\alpha^{\frac{1}{\kappa_\alpha}} \alpha_{S,t}(i)^{\frac{\kappa_\alpha - 1}{\kappa_\alpha}} + (1-\zeta_\alpha)^{\frac{1}{\kappa_\alpha}} \alpha_{L,t}(i)^{\frac{\kappa_\alpha - 1}{\kappa_\alpha}}\right]^{\frac{\kappa_\alpha}{\kappa_\alpha - 1}} \tag{3.5}$$

where ζ_α is the share of short-term assets in the aggregate portfolio and κ_α is the elasticity of substitution between short- and long-term assets. When $\zeta_\alpha \to \infty$, short- and long-term assets are perfect substitutes and changes in the relative supply of long-term assets (due to an asset purchase program) do not produce real effects.

In turn, the short-term portfolio is itself a CES aggregate of short-term domestic bonds, $B_{HS,t}$, and short-term foreign bonds, $B_{FS,t}$, both in non-zero net supply:

$$\alpha_{S,t}\left(i\right) = \left[\zeta_S^{\frac{1}{\kappa_S}} \left(\frac{B_{HS,t}(i)}{P_t}\right)^{\frac{\kappa_S - 1}{\kappa_S}} + (1-\zeta_S)^{\frac{1}{\kappa_S}} \left(\frac{B_{FS,t}(i)}{P_t}\right)^{\frac{\kappa_S - 1}{\kappa_S}}\right]^{\frac{\kappa_S}{\kappa_S - 1}} \tag{3.6}$$

where P_t is the aggregate price level, ζ_S is the share of domestic short-term bonds in the sub-portfolio and κ_S is the elasticity of substitution between domestic and foreign short-term bonds.

Similarly, the long-term portfolio is also a CES aggregate of long-term domestic bonds, $B_{HL,t}$, and long-term foreign bonds, $B_{FL,t}$.

$$\alpha_{L,t}\left(i\right) = \left[\zeta_L^{\frac{1}{\kappa_L}} \left(\frac{q_{L,t} B_{HL,t}(i)}{P_t}\right)^{\frac{\kappa_L - 1}{\kappa_L}} + (1-\zeta_L)^{\frac{1}{\kappa_L}} \left(\frac{\psi_t q^*_{L,t} B_{FL,t}(i)}{P_t}\right)^{\frac{\kappa_L - 1}{\kappa_L}}\right]^{\frac{\kappa_L}{\kappa_L - 1}} \tag{3.7}$$

where ζ_L is the share of domestic long-term bonds in the sub-portfolio and κ_L is the elasticity of substitution between domestic and foreign long-term bonds. We model long-term bonds following Woodford (2001). Specifically, a long-term bond has a payment structure ρ^{T-t-1} for $T > t$ and $0 \leq \rho \leq 1$. The value of this type of instrument issued in period t, in any future period $t+j$, is $q^{-j}_{L,t+j} = \rho^j q_{L,t+j}$, where parameter ρ captures the maturity of the long-term bonds.[1] Foreign long-term bonds share the same properties and have price $q^*_{L,t}$. Finally, $\psi_t = P^*_t / P_t$ denotes relative prices.

[1]When $\rho = 0$ this asset collapses to a one period bond, while for $\rho = 1$ this asset resembles a console.

3.1.3 First order conditions

Household i chooses sequences $\{C_T(i), B_{HS,T}(i), B_{FS,T}(i), B_{HL,T}(i), B_{FL,T}(i)\}_{T=t}^{\infty}$ to maximize 3.1 subject to 3.4 and the relevant no-Ponzi game constraints, given initial bond holdings, $B_{HS,t-1}(i)$, $B_{FS,t-1}(i)$, $B_{HL,t-1}(i)$, $B_{FL,t-1}(i)$. The first order conditions with respect to $B_{HS,t}(i)$ and $B_{HL,t}(i)$ read as follows:

$$\frac{1}{C_t} = \beta E_t\left[\frac{R_t}{C_{t+1}\pi_{t+1}}\right] + \frac{1}{C_t}\psi\left(\zeta_\alpha\frac{\alpha_t}{\alpha_{S,t}}\right)^{\frac{1}{\kappa_\alpha}}\left(\zeta_S\frac{\alpha_{S,t}}{b_{HS,t}}\right)^{\frac{1}{\kappa_S}} \tag{3.8}$$

$$\frac{q_{L,t}}{C_t} = \beta E_t\left[\frac{R_{L,t+1}q_{L,t+1}}{C_{t+1}\pi_{t+1}}\right] + \frac{1}{C_t}\psi\left((1-\zeta_\alpha)\frac{\alpha_t}{\alpha_{L,t}}\right)^{\frac{1}{\kappa_\alpha}} q_{L,t}\left(\zeta_L\frac{\alpha_{L,t}}{q_{L,t}b_{HL,t}}\right)^{\frac{1}{\kappa_L}} \tag{3.9}$$

Similar expressions hold for $B_{FS,t}(i)$ and $B_{FL,t}(i)$. The last two terms in expressions (3.8) and (3.9) capture the imperfect substitutability between short- and long-term bonds. As we show below, these drive the term premium, which depends on the relative holdings of bonds in households' portfolios, as well as on the elasticities of substitution between maturities.[2] By log-linearizing (3.8) and (3.9) we obtain the following two expressions

$$0 = \beta\frac{R}{\pi}\left(\hat{R}_t - E_t\hat{\pi}_{t+1} - E_t\hat{C}_{t+1} + \hat{C}_t\right) + \left(1-\beta\frac{R}{\pi}\right)\left(\frac{1}{\kappa_\alpha}\left(\hat{\alpha}_t - \hat{\alpha}_{S,t}\right) + \frac{1}{\kappa_S}\left(\hat{\alpha}_{S,t} - \hat{b}_{HS,t}\right)\right) \tag{3.10}$$

$$\begin{aligned} 0 = {} & \beta\tfrac{R}{\pi}\left(E_t\hat{R}_{L,t+1} + E_t\hat{q}_{L,t+1} - E_t\hat{\pi}_{t+1} - E_t\hat{C}_{t+1} - \hat{q}_{L,t} + \hat{C}_t\right) \\ & + \left(1-\beta\tfrac{R}{\pi}\right)\left(\tfrac{1}{\kappa_\alpha}\left(\hat{\alpha}_t - \hat{\alpha}_{L,t}\right) + \tfrac{1}{\kappa_L}\left(\hat{\alpha}_{L,t} - \hat{q}_{L,t} - \hat{b}_{HL,t}\right)\right) \end{aligned} \tag{3.11}$$

3.2 Term premia

Imperfect substitutability between short- and long-term assets gives rise to risk premia in both regions. Combining the log-linearized Euler equations (3.10) and (3.11), with the log-linearized long-term rate, $\hat{R}_{L,t}$, we obtain the following expression:

$$\frac{R_L\widehat{R}_{L,t} - \rho E_t\widehat{R}_{L,t+1}}{R_L - \rho} = \widehat{R}_t + \left(\frac{\pi}{\beta R} - 1\right)\hat{T}_t \tag{3.12}$$

where

$$\hat{T}_t = \left(\frac{1}{\kappa_\alpha}\right)\left(\hat{\alpha}_{L,t} - \hat{\alpha}_{S,t}\right) + \left(\frac{1}{\kappa_S}\right)\left(\hat{\alpha}_{S,t} - \hat{b}_{HS,t}\right) - \left(\frac{1}{\kappa_L}\right)\left(\hat{\alpha}_{L,t} - \hat{q}_{L,t} - \hat{b}_{HL,t}\right) \tag{3.13}$$

Equation (3.12) shows the risk-adjusted returns across assets of different maturities. Namely, the one-period return from holding a domestic long-term bond is equal to the return from holding a short-term bond scaled by a premium that households require for deviating from steady-state portfolio holdings. By iterating on

[2] Note that the CES specification on bond holdings excludes corner solutions. That is, there will always be a well-defined term premium between short- and long-term assets. Alternative specifications of portfolio adjustment costs can allow for a zero term-premium (see Alpanda and Kabaca (2019) and Chen *et al.* (2012) and references therein).

equation (3.12) we obtain an expression for long-term yields:

$$\hat{R}_{L,t} = \left(1 - \frac{\rho}{R_L}\right) E_t \sum_{s=0}^{\infty} \left(\frac{\rho}{R_L}\right)^s \left[\hat{R}_{t+s} + \left(\frac{\pi}{\beta R} - 1\right) \hat{T}_{t+s}\right]. \tag{3.14}$$

Expression (3.14) shows that the long-term rate is a function of expected short-term rates and a term premium. The latter is a function of the short- and long-term portfolios (equation 3.13), suggesting that the term premium and consequently the long-term yield is higher as households hold more long-term relative to short-term bonds.

Besides the elasticity of substitution between short and long-term assets, κ_α, given that households also buy foreign long-term bonds, the elasticity of substitution between home and foreign long-term bonds, κ_L, also affects the term premium. In the extreme case of perfect substitution (i.e. $\kappa_\alpha, \kappa_S, \kappa_L \to \infty$), the term premium falls to zero and the long-term rate is simply the sum of expected short-term rates. Notice that the dynamic behavior of the term premium is determined by the holdings of short-term bonds relative to the long-term portfolio as well as the holdings of home long-term bonds relative to the long-term portfolio. Therefore, the endogenous structure of the term premium captures the effects of changes in the supply of long-term bonds engineered by the QE program of the union wide central bank. If QE involves purchases of Periphery long-term bonds, then the Periphery will face a drop in the supply of its long-term debt, $\hat{b}_{HL,t}$, which lowers the term premium and the interest rate on those assets, ceteris paribus. The lower the substitutability between home and foreign long-term assets, the larger the drop in the term premium.

Moreover, the endogenous structure of the term premium allows analyzing how household behavior may mitigate the effects of QE. For a given drop in the supply of Periphery long-term bonds, the drop in the term premium may be partly offset by a switch of domestic households to foreign long-term bonds. In contrast, if the households shift their portfolio to short-term assets the drop in the term premium is further enhanced. In other words, the nature of portfolio rebalancing taking place following a QE program can work either in favor or against the intended effects on real rates.

3.2.1 Arbitrage between home and foreign bonds

We now discuss the relationships between home and foreign yields at each maturity. As a result of limits to arbitrage, the home and foreign yields differ from each other as follows:

$$\widehat{R}_t - \widehat{R}_t^U = \left(\frac{\pi}{\beta R} - 1\right) \frac{1}{\kappa_S} \left(\widehat{b}_{HS,t} - \left(\widehat{\psi}_t + \widehat{b}_{FS,t}\right)\right)$$

$$\frac{R_L \widehat{R}_{L,t} - \rho E_t \widehat{R}_{L,t+1}}{R_L - \rho} - \frac{R_L \widehat{R}^*_{L,t} - \rho E_t \widehat{R}^*_{L,t+1}}{R_L - \rho} = \left(\frac{\pi}{\beta R} - 1\right) \frac{1}{\kappa_L} \left[(\widehat{q}_{L,t} + \widehat{b}_{HL,t} - \widehat{\psi}_t) - (\widehat{q}^*_{L,t} + \widehat{b}_{FL,t})\right]$$

The no-arbitrage condition with respect to short-term bonds is obtained by combining the FOCs with respect to $B_{HS,t}(i)$ and $B_{FS,t}(i)$. When $\kappa_S \to \infty$, short rates are equalized in the model for every period. When calibrating κ_S we assume it takes a relatively high value so that the central bank can set short rates across the union in both Periphery and Core economies.

The second equation similarly shows the arbitrage between home and foreign bonds in long-term maturities, and is obtained by using the FOCs with respect to $B_{HL,t}(i)$ and $B_{FL,t}(i)$. Note that the left-hand-side of this equation indicates the one-period holding return differentials between home and foreign long-term bonds. Similar to the arbitrage in short-term maturities, the differential here is governed by the elasticity parameter between home and foreign bonds in the long-term sub-portfolio, κ_L. In the next sections we will calibrate κ_L (as well as κ_a) to match observed changes in portfolios following ECB purchases.

The above equations can be used to obtain a relationship between $\hat{T}_t$ and $\hat{T}_t^*$, which governs the link between domestic and foreign term premiums:

$$\hat{T}_t = \hat{T}_t^* + \frac{1}{\kappa_L}\left[(\widehat{q}_{L,t} + \widehat{b}_{HL,t} - \widehat{\psi}_t) - (\widehat{q}^*_{L,t} + \widehat{b}_{FL,t})\right]$$

assuming $\kappa_S \to \infty$. This expression implies that if the supply of home bonds increases relative to that of foreign bonds in long-term maturities, households would require a higher domestic premium. Note that when long-term bonds are perfect substitutes ($\kappa_L \to \infty$), the home term premium cannot deviate from foreign premium.

3.2.2 Aggregate demand

How do the financial returns affect domestic demand? In order to answer this question, we combine the FOCs with respect to all assets as well as marginal utility of consumption to obtain:

$$\widehat{c}_t = \widehat{c}_{t+1} - (PR_t - E_t \widehat{\pi}_{t+1})$$

$$PR_t = \zeta_a \zeta_S \widehat{R}_t + (1-\zeta_a)\zeta_L\left(\widehat{R}_t + \widehat{T}_t\right) + \zeta_a(1-\zeta_S)\widehat{R}_t^U + (1-\zeta_a)(1-\zeta_L)\left(\widehat{R}_t^U + \widehat{T}_t^*\right)$$

PR_t denotes the portfolio return weighted by portfolio shares. The above expression implies a higher impact from foreign yields as households hold more foreign bonds in their portfolios. Although conventional monetary policy, that is an increase in short-term rates, produces the standard effect of disincentivizing consumption, here, purchases of long-term bonds by the union-wide central bank generate an additional effect through the term premium. Namely, QE will lower the long-term yield component of the term premium and strengthen contemporaneous consumption demand.

Finally, note that when all assets are perfectly substitutable, all asset classes will yield the same return.

Thus, when $\kappa_\alpha, \kappa_S, \kappa_L \to \infty$, portfolio components in the modified arbitrage conditions disappear; therefore. $\widehat{R}_t = \widehat{R}_t^U = \widehat{R}_t + \widehat{T}_t = \widehat{R}_t^U + \widehat{T}_t^*$. In this frictionless world, the model implies the same Euler condition as in the basic New Keynesian model: $\widehat{c}_t = \widehat{c}_{t+1} - (R_t - E_t\widehat{\pi}_{t+1})$. In the absence of this setting, monetary policy is conventional and changes in consumption demand are influenced through changes in the short-term interest rate alone.

3.3 Firms

The production side in both regions consists of monopolistically competitive firms, who use a linear technology, employing labor, in order to produce a final good. Each firm sets the price of its good infrequently, as in Calvo (1983), regardless of the market in which the good is sold.[3]

At each date, each firm changes its price with a probability $1 - \omega$. The price level for domestic goods at date t is defined as

$$P_{H,t} = \left[\omega P_{H,t-1}^{1-\theta} + (1-\omega)\widetilde{p}_t(h)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{3.15}$$

Each brand h is produced by a single firm following the linear technology

$$Y_t(h) = A_t L_t(h) \tag{3.16}$$

where A_t is a country specific productivity shock, which follows the log stationary AR(1) process, $a_t = \rho_a a_{t-1} + \varepsilon_{a,t}$. Firms set their prices by maximizing their expected discounted profits

$$\max_{p_{t+\varsigma}} E_t \sum_{\varsigma=0}^{\infty} \omega^{\varsigma} \Lambda_{t,t+\varsigma} \left\{p_{t+\varsigma}(h)\left(y_{t+\varsigma}(h) + y_{t+\varsigma}^*(h)\right) - (1-\tau)W_{t+\varsigma}^h L_{t+\varsigma}^h(h)\right\} \tag{3.17}$$

where $\Lambda_{t,t+\varsigma} = \beta^{\varsigma}\left(C_{t+\varsigma}/C_t\right)^{-\sigma}\left(P_{t+\varsigma}/P_t\right)$ is the stochastic discount factor and τ is a labor subsidy.[4] The firm maximizes (3.17) subject to (3.16) and domestic, $y_{t+\varsigma}(h)$, and foreign demand, $y_{t+\varsigma}^*(h)$, for its product. The optimal price for the domestically produced good in the home and foreign country is specified as

$$\tilde{p}_t(h) = \frac{\theta}{\theta-1}\frac{E_t \sum_{\varsigma=0}^{\infty} \omega^{\varsigma}\Lambda_{t,t+\varsigma} MC_{t+\varsigma}\left(y_{t+\varsigma}(h) + y_{t+\varsigma}^*(h)\right)}{E_t \sum_{\varsigma=0}^{\infty} \omega^{s} Q_{t,t+\varsigma}\tilde{c}_{t+\varsigma}(h)} \tag{3.18}$$

3.3.1 Aggregate prices

The aggregate consumption price index for the home country is specified as

$$P_t = \left[\delta(P_{H,t})^{1-\rho} + (1-\delta)P_{F,t}^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{3.19}$$

[3]In the absence of a currency union, this is equivalent to producer currency pricing.

[4]The optimal subsidy removing the distortion from monopolistic competition satisfies $(1-\tau)(\frac{\theta}{\theta-1}) = 1$ and renders the steady state and the flexible price equilibrium efficient.

where P_H and P_F are price indices for home and foreign goods, expressed in the domestic currency and defined as

$$P_{H,t} = \left[\left(\frac{1}{n}\right)\int_0^n p_t(h)^{1-\theta}dh\right]^{\frac{1}{1-\theta}}, \; P_{F,t} = \left[\left(\frac{1}{1-n}\right)\int_n^1 p_t(f)^{1-\theta}df\right]^{\frac{1}{1-\theta}} \tag{3.20}$$

3.4 Fiscal policy

Each country has an independent fiscal authority. The fiscal authority in each region uses lump-sum taxes/transfers from the central bank to finance expenditure and to stabilize debt. The government budget constraint in the home country is specified as follows:

$$B_{S,t} + q_{L,t}B_{L,t} + Z_t + \tau_t^w W_t L_t = R_{t-1}B_{S,t-1} + q_{L,t}R_{L,t}B_{L,t-1} + P_{H,t}G_t \tag{3.21}$$

where $B_{S,t}$ and $B_{L,t}$ represent the stocks of total short- and long-term debt respectively, while Z_t represents transfers by the central bank. Expressing the government budget constraint in real terms we receive the following expression:

$$b_{S,t} + q_{L,t}b_{L,t} + z_t + \tau_t^w w_t L_t = \frac{R_{t-1}b_{S,t-1}}{\Pi_t} + \frac{q_{L,t}R_{L,t}b_{L,t-1}}{\Pi_t} + p_{H,t}g_t \tag{3.22}$$

where $b_{S,t} = B_{S,t}/P_t$, $b_{L,t} = B_{L,t}/P_t$, and $\Pi_t = P_t/P_{t-1}$ is gross inflation.

To determine long-term debt we assume that when financing the deficit the government is indifferent between issuing short-term or long-term bonds. This implies the following rule for the issuance of long-term bonds

$$q_{L,t}B_{L,t} = \gamma B_{S,t}$$

which specifies that long-term bonds are issued at a fraction γ of short-term bonds.

Finally, government spending follows an exogenous stationary $AR(1)$ process and the law of motion of tax revenues is assumed to take the form:

$$\tau_t^w = \Xi\left(\frac{b_{S,t-1} + q_{L,t}b_{L,t-1}}{b_S + q_L b_L}\right)^{\tau_b} \tag{3.23}$$

3.5 Union-wide quantitative easing

When the central bank does not engage in quantitative easing, monetary policy entails adjusting the nominal interest according to a Taylor-type rule targeting average inflation in the union, $\widehat{\pi}_t^U = n\widehat{\pi}_t + (1-n)\widehat{\pi}_t^*$. The interest rate rule of the central bank is specified as follows:

$$\log R_t^U = \rho \log R_{t-1}^U + (1-\rho)\left(\log R + r_\pi \log \frac{\pi_t^U}{\pi}\right) + \varepsilon_{r,t} \tag{3.24}$$

Alternatively, when the central bank engages in a program of quantitative easing it buys a fraction υ_t of total long-term bonds issued in the union.

$$q_{L,t}B^{cb}_{L,t} = s\upsilon_t(q_{L,t}B_{L,t} + q^*_{L,t}B^*_{L,t}). \tag{3.25}$$

For simplicity, we assume that the policy variable υ_t of the central bank follows an $AR(1)$ stationary process $\upsilon_t = \rho_\upsilon \upsilon_{t-1} + \epsilon_{\upsilon,t}$ and a positive shock to this variable represents the QE shock. We assume a zero mean process for υ_t to ensure that there is no QE at the steady state. The allocation of purchases across countries is determined by s, which reflects the fraction of long-term bonds issued in the Periphery. Conversely, the fraction $1 - s$ reflects the share of long-term bonds purchased from the Core. This is in line with the practice of the ECB, whose QE program is designed according to pre-determined GDP and population weights across countries.

In order for quantitative easing to be neutral on fiscal balances, i.e. to circumvent the financing of the program via taxes from households, we assume that long-term bond purchases are financed with issuance of new short-term debt by the central bank:

$$\begin{aligned} q_{L,t}B^{CB}_{L,t} &= B^{CB}_{S,t} \\ q^*_{L,t}B^{*CB}_{L,t} &= B^{*,CB}_{S,t} \end{aligned}$$

The net receipts from QE, Z_t, which are then transferred back to the government are defined then defined as in Harrison (2017):

$$Z_t = q_{L,t}R_{L,t}B^{CB}_{L,t-1} - q_{L,t}B^{CB}_{L,t} + B^{CB}_{S,t} - R^U_t B^{CB}_{S,t-1}$$

Following the above specification of QE, the equilibrium in the market for peripheral long-term debt satisfies:

$$B_{HL,t} + B^*_{HL,t} + B^{cb}_{L,t} = B_{L,t}. \tag{3.26}$$

Substituting equation (3.25) into (3.26) yields:

$$B_{L,t} = \frac{B_{HL,t} + B^*_{HL,t} + s\upsilon_t \frac{q^*_{L,t}B^*_{L,t}}{q_{L,t}}}{1 - s\upsilon_t} \tag{3.27}$$

The respective expression for the Core reads as follows:

$$B^*_{L,t} = \frac{B_{FL,t} + B^*_{FL,t} + (1-s)\upsilon_t \frac{q_{L,t}B_{L,t}}{q^*_{L,t}}}{1 - (1-s)\upsilon_t} \tag{3.28}$$

From the two expressions above it is clear that, for a given level of long-term debt $\overline{B}_L$ and $\overline{B}_L^*$, a QE shock tends to decrease private holdings of long-term debt, $B_{HL,t} + B_{HL,t}^*$ and $B_{FL,t} + B_{FL,t}^*$, in both regions. Therefore, the QE shock triggers a portfolio rebalancing in the union by removing long-term debt from the private sector and triggering a shift towards shorter maturities. As discussed the effect of portfolio rebalancing will tend to be expansionary.[5] However, since each region's long-term debt is held by both domestic and foreign households, a QE shock does not necessarily imply that both households reduce their holdings of long-term debt. The extent to which they do so depends, first, on the elasticities of substitution between long- and short-term assets, κ_α and, second, on the capital key weights, s and $1 - s$. The lower s is, the weaker the extent of portfolio rebalancing and hence the smaller the drop in the term premium. This is an additional source of asymmetry in our model, apart from heterogeneity in financial frictions in portfolios, which can lead to differential effects of QE in the monetary union.

4 Quantitative Analysis

In this section we perform a quantitative analysis of the model to investigate the effects of portfolio rebalancing in the union.

4.1 Calibration

We calibrate the economy by assuming that parameters unrelated to quantitative easing are common across regions, but a smaller number of parameters that relate to portfolio shares and elasticities can vary across the Periphery and Core. The values for parameters related to portfolio elasticities are set to replicate the impulse responses from an SVAR on the Periphery and Core identified using sign restrictions on the movements of bond holdings. The values of all parameters can be seen in Table 2.

4.1.1 Parameters constant across regions

The discount factor β is set to 0.99, which implies a a steady-state nominal interest rate of 1.01. The parameter defining the level of the household asset portfolio, ψ, is calibrated to match a steady-state nominal interest rate of 1.01. The inverse of the elasticity of labor supply, γ, is set to 0.5, a standard value in the macroeconomic literature. The parameter reflecting price rigidity ω is set to 0.9. The responsiveness of the nominal interest rate to average union inflation, ϕ_π, is set to 2.038, and the smoothing parameter, ρ_R, to 0.8394. Both values are consistent with the calibration of the ECB's quantitative macro model of the Euro Area, EAGLE. Regarding

[5] We assume that long-term debt, $B_{L,t}$ and $B_{L,t}^*$ is fixed in each region. Note that there have been recipient countries which increased their issuances of long-term debt while QE was active in the euro area. Clearly, these additional issuances undo the intended effects of QE. We abstract from this scenario as our objective is to restrict attention to the pure portfolio rebalancing effects of QE which may be also subject to structural asymmetries.

the elasticity parameter of tax policy, τ_b, we set it to 0.9 in order to ensure stability of the model.

4.1.2 Parameters different across regions

The share of long-term bonds purchased from the periphery s is set to 0.4 in line with the relative size of the population in the union. This also implies that the region size, n is equal to 0.4. Conversely, $1-s$ and $1-n$ is equal to 0.6. The coupon rate of long-term bonds, ρ, is calibrated to match average duration of long-term bonds in each region and is set at 0.9701 in the Periphery and 0.9695 in the Core. Regarding home bias in the consumption of goods, λ is calibrated 0.5 in order to match an import ratio of 0.2525 in the Periphery, which is close to the value found in EAGLE and the European Commission's macro model, QUEST.

Short- and long-term bond shares in households' portfolios are calibrated using data from the ECB's Security Holdings Statistics. The share of domestic short-term bonds to foreign short-term bonds ζ_S is 0.95 in the Periphery and 0.72 in the Core. The share of domestic long-term bonds to foreign long-term bonds ζ_L is 0.97 in the Periphery and 0.83 in the Core. The shares that govern the more general weighting of short-term and long-term bonds in households' portfolios ζ_a is 0.48 in the Periphery and 0.67 in the Core. Notably, the definition of short-term bonds includes holdings of the monetary base for each region.

4.1.3 VAR-based approach for the estimation of portfolio elasticities

We place empirical discipline on the elasticities of assets of different maturity and location by estimating an SVAR for the Periphery and Core. More specifically, using the data from the ECB's Security Holdings Statistics, we identify a quantitative easing shock in each region by estimating an SVAR over 2013Q4 to 2018Q4, identified using sign restrictions.

For each region, the objective is to estimate the following system of equations:

$$AY_t = \sum_{k=1}^{K} C_k Y_{t-k} + Bu_t \tag{4.1}$$

where Y_t is a vector of endogenous variables (e.g. short-term bond holdings, GDP, and other endogenous variables) for a given quarter t a. C_k is a matrix of the own- and cross-effects of the k^{th} lag of the variables on their current observations. B is a diagonal matrix so that u_t is a vector of orthogonal i.i.d. shocks to government expenditures such that $Eu_t = 0$ and $E\left[u_t u_t^{'}\right] = I$. A is a matrix that allows for contemporaneous effects between the endogenous variables in Y_t.

The specification is estimated in logs using an OLS regression. We employ 2 lags of the endogenous variables given that our data is short. OLS provides an estimate for the matrices $A^{-1}C$, but additional identification assumptions are necessary to estimate the coefficients in A and B.

Identifying a quantitative easing shock

Y_t contains the variables: *GDP, CPI deflator, total debt, short debt, long debt. Total debt* is defined as the share of short-term debt to long-term debt. *Short-term debt* is defined as the share of domestic short-term debt to foreign long-term debt. And, *long-term debt* is defined as the ratio of domestic long-term debt to foreign long-term debt.

We impose non-recursive short-run restrictions to identify exogenous variations in the bonds held outright, which are referred to as the QE shock. Our framework is similar to that of Gambacorta (2014), Bhattarai et al. (2018) and Gambetti and Musso (2017) who focus on the macroeconomic implications of QE using a central bank balance sheet variable as an instrument of policy. Our methodology is also related to Wright (2012), Baumeister and Benati (2013) and Bluwstein and Canova (2016). The latter use an agnostic approach for identification and estimate spillover effects of unconventional monetary policy by the European Central Bank to non-Euro area countries in Europe.

Table 1 describes the identifying restrictions. The columns correspond to the variables while the rows correspond to the shock intended to be identified. The first two shocks (Supply and Demand) are shocks related affecting the real economy, determining slow-moving variables like output and prices. The last two shock (Financial and QE) are, respectively, shocks to long-term interest rates quantitative easing. The financial shock includes restrictions that the long-term interest rate adjusts contemporaneously to changes in output, prices, and asset purchases by the ECB. For the QE shock, we assume that the monetary policy instrument reacts contemporaneously only to the long-term interest rate. The assumption that the ECB does not react contemporaneously to GDP and prices is because the ECB cannot immediately observe these variables. We thus assume that the QE policy of the ECB is well approximated by a rule that determines the ECB's purchase of bonds as a linear function of the contemporaneous long-term yield and the lags of macroeconomic and financial variables. Any unanticipated non-systematic variations in the bonds held outright are then identified as a shock to the QE policy that is exogenous to the state of the economy. This approach is analogous to that for the identification of monetary policy shocks in the conventional monetary policy analysis as in e.g., Leeper, Sims, and Zha (1996) and Sims and Zha (2006a; 2006b) to identify conventional monetary policy shocks in the US.

Table 1: Identification Restrictions

	Supply shock	Demand shock	Financial shock	QE shock
GDP	+	+	+	?
CPI deflator	?	+	+	-
Total debt	?	?	?	+
Short term debt	?	?	?	?
Long term debt	?	?	+	?
10-year govt. bond yield	?	?	-	-

Notes: Rows denote the variables in the VAR. Columns denote the identified shocks. "?" denotes unrestricted contemporaneous effect. Sign restrictions are imposed for 1 quarter.

Estimation of the SVAR for each region yields the following results. Figure 4.1 plots the impulse responses for the Periphery while Figure 4.2 for the core. We have scaled the responses of the variables to be consistent with a QE shock of size 10% of EA GDP, as in our quantitative model. In both regions, GDP and inflation increase following a QE shock, but the effect is larger for the Periphery. This is consistent with the larger decline in the 10-year government bond yield for the Periphery. At the same time, total debt increases by more in the Periphery, and long-term debt declines by more in the Core. The results suggest that financial frictions are higher in the Periphery so for a given size of QE shock across regions Periphery portfolios rebalance to a larger degree than in Core.

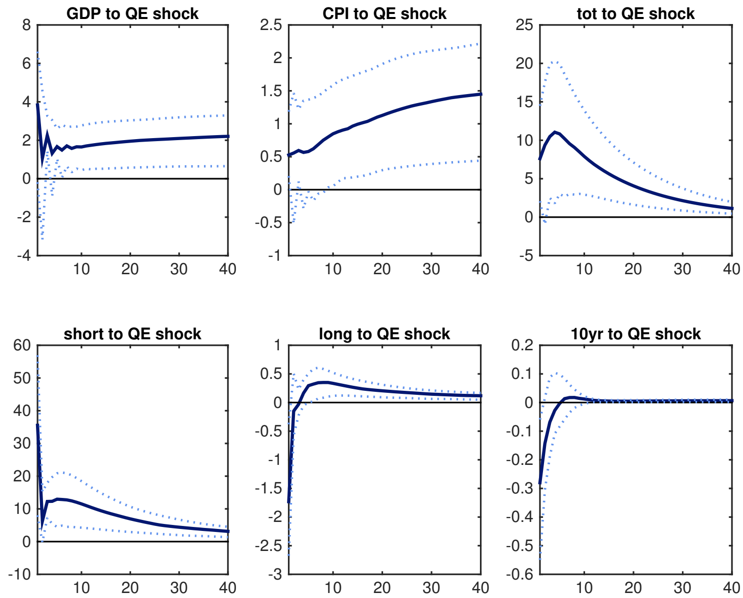
Figure 4.2 plots a weighted average of the impulse response following a QE shock for the EA as a whole, where the weights reflect the capital key by the ECB, 40% of long-term bond are purchased from the Periphery, and the remaining 60% from the Core.

4.1.4 Matching impulse responses in the quantitative model and in the data

Given these response from the QE shock in the data we then perform the following exercise. We set values for the elasticities of substitution across assets of different maturity and origin so that our quantitative model yields the same impulse responses as those in the data.

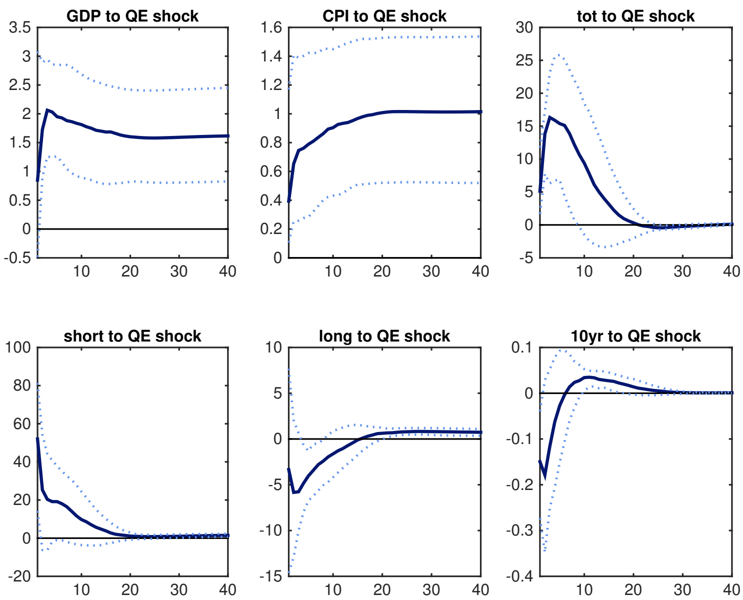
The elasticity with respect to domestic short-term bonds and foreign short-term bonds, κ_S, is calibrated to a high value so that the union-wide central can set common short rates across the union. So, $\kappa_S = \kappa_S^* = 100$. The elasticities with respect to short-term bonds and long-term bonds, κ_a, κ_a^*, and with respect to domestic long-term bonds and foreign short-term bonds, κ_L, κ_L^*, are set so that the change in portfolio shares and the aggregate union-wide term premium are close to the changes observed in the data. In particular, we calibrate them to match two targets i) a decline in long-term debt for the EA by around 4% ii) an increase in total debt for the EA by around 15%, and iii) a decline in long-term interest rates by around 20 basis points for the EA. This term premium decline is consistent with evidence from DSGE based studies of quantitative easing in the EA (see Hohberger *et al.* (2019) and references therein).

Figure 4.1: SVAR: QE shock in the Periphery



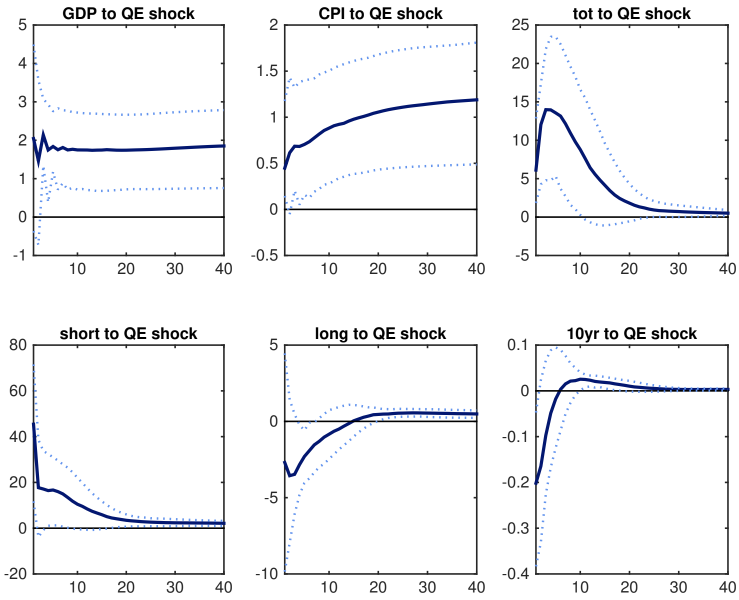
Notes: SVAR identified with sign restrictions. (Scaled) impulse response functions in the Periphery to a quantitative easing shock of size 10% of EA GDP.

Figure 4.2: SVAR: QE shock in the Core



Notes: SVAR identified with sign restrictions. (Scaled) impulse response functions in the Core to a quantitative easing shock of size 10% of EA GDP.

Figure 4.3: SVAR: QE shock in the EA



Notes: SVAR identified with sign restrictions. (Scaled) impulse response functions in the EA (weighted responses of Periphery and Core using $s = 0.4$) to a quantitative easing shock of size 10% of EA GDP.

Table 2: Parameters

	Symbol	Periphery	Core
Inflation target (gross, qtr.)	π	1.005	1.005
Discount factor	β	0.99	0.99
Country size	n	0.4	0.6
Capital key	s	0.4	0.6
Habits	h	0.75	0.75
Portfolio shares - short vs. long portfolio	ζ_a	0.48	0.67
- short domestic vs. foreign bonds	ζ_S	0.95	0.72
- long domestic vs. foreign bonds	ζ_L	0.97	0.83
Portfolio elasticities - short vs. long portfolio	κ_a	0.22	0.65
- short domestic vs. foreign bonds	κ_S	100	100
- long domestic vs. foreign bonds	κ_L	0.1	0.2
Portfolio level in utility	ψ	0.005	0.005
Inverse of the elasticity of labor supply	γ	0.5	0.5
Coupon rate for long-term bonds	ρ	0.9701	0.9695
Home bias in goods	λ	0.5	0.49
Gross markup - domestic goods price	$\frac{\theta}{\theta-1}$	1.25	1.25
Calvo price rigidity	ω	0.9	0.9
Taylor rule - persistence	ρ_R	0.894	0.894
- Inflation sensitivity	ϕ_π	2.038	2.038
Tax rate	τ_w	0.2554	0.2485
Elasticity in tax policy	τ_b	0.9	0.9

4.1.5 Steady-state ratios

The steady-state of the model is calibrated as follows.

Table 3: steady s

(relative to output)	Symbol	Periphery	Core
Consumption	c/y	0.7853	0.8072
Gov. expenditure	g/y	0.20	0.20
Exports[a]	y_f^*/y	0.2778	0.1296
Imports[a]	y_f/y	0.2631	0.1368
Bond supply / GDP (ann.)			
short	b_S/y	0.94	0.76
long	$q_L b_L/y$	0.56	0.43
Bond holdings / GDP (ann.)			
short home	b_{HS}/y	0.35	0.75
long home	$q_L b_{HL}/y$	0.38	0.0764
short foreign	b_{FS}/y	0.02	0.29
long foreign	$q_L^* b_{FL}/y$	0.01	0.09

4.2 Quantitative easing shock

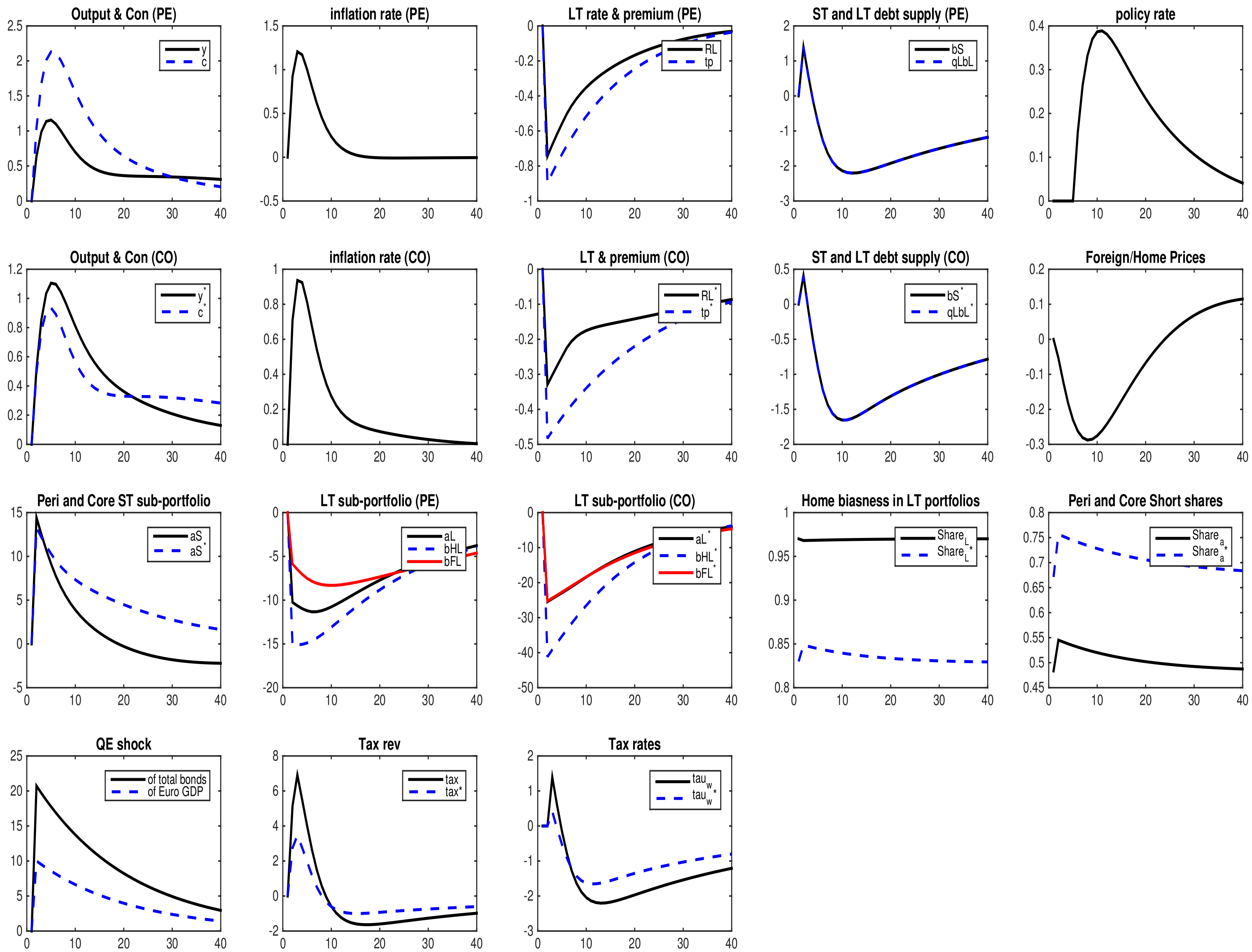
We calibrate the size of the quantitative easing shock to be 10% of EA GDP. This size is in line with the announcement of the ECB in January 2015, when the program was originally implemented. Recall that this total volume will be allocated across countries according to the parameter s, which reflects the ECB capital key, and attaches weights based on relative regional GDP and population size.

In our model, the effects of long-term bond purchases affect asset prices in two key ways. First, purchases of long-term bonds imply a direct reduction of the term premium in each region (i.e. a flattening of the yield curve). Second, the effects of reduction in term premiums spill over through portfolio rebalancing of cross-border assets within the union. The results can be illustrated in Figure 4.4, which plots impulse responses form a 10% quantitative easing shock, that is purchases of $B_{L,t}^{CB}$ by the union-wide central bank, financed with newly issued short-term debt.

The magnitude of these effects is driven by the extent to which assets are imperfectly substitutable in the model. The extent of this imperfect substitutability is governed by the relative difference in parameters $\kappa_a, \kappa_S, \kappa_L$ and $\kappa_a^*, \kappa_S^*, \kappa_L^*$. Given the calibrated portfolio elasticities, which have been set to target the four moments discussed above (in particular a union-wide decline in the term premium of 65 bp), a 10% QE shock, generates a larger drop in the term premium in the Periphery (75bp) than in the Core (30bp).

This in turn implies a more expansionary effect on Periphery consumption than in that of the Core. Interestingly, this is despite the weight placed on the purchases of Periphery assets ($s = 0.4$). Consumption in the Periphery rises by approximately 2% whereas by 0.9% in the Core. The hump-shaped response of consumption, and in extension of output, follows from habit persistence.

Figure 4.4: Quantitative easing shock



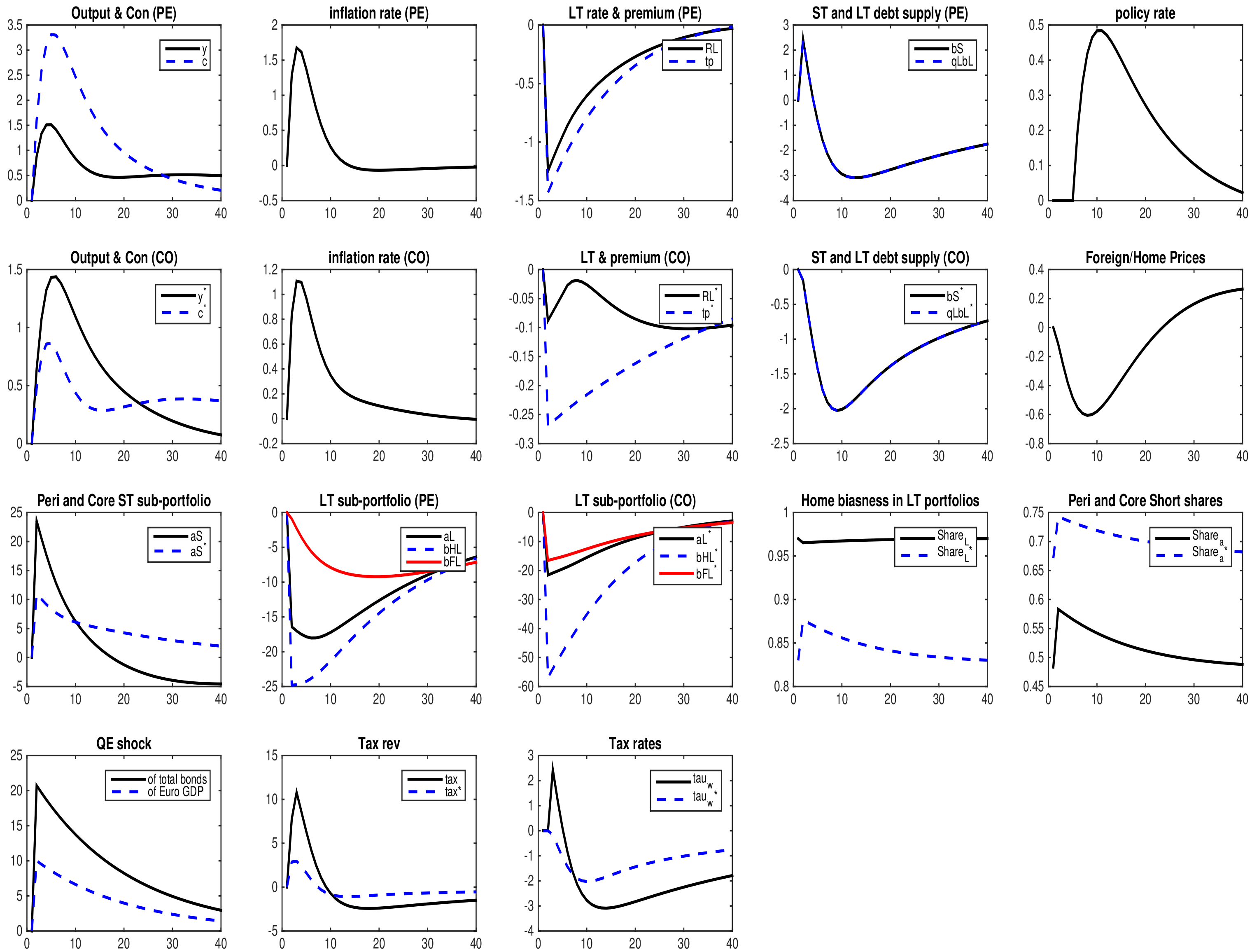
Notes: Impulse response functions to a quantitative easing shock of size 10% of union GDP (in % deviations from steady-state).

4.3 Sensitivity to capital key $s = 0.6$

What if the union-wide central bank targeted purchases of long-term assets across regions differently? In the previous discussion we assumed that the central bank purchased long-term debt across regions according to the capital key, that is a weight reflecting relative differences in population and GDP across regions. In Figure 4.5 we relax this assumption and assume a value of $s = 0.6$. This implies that a greater share of long-term bonds will be purchased by the Periphery relative to the Core.

The results are in line with the intuition. The term premium in the Periphery is now reduced by more (1.3bp instead of 75bp) and the term premium in the Core is reduced by less (10bp instead of 30bp). The effects are now more expansionary in the Periphery (consumption increases by 3.2% at the peak) and less expansionary in the Core (consumption increases by less than 0.75% at the peak).

Figure 4.5: Quantitative easing shock, sensitivity to capital key



Notes: Impulse response functions to a quantitative easing shock of size 10% of union GDP. Sensitivity to capital key ($s = 0.6$) (in % deviations from steady-state).

5 Optimal Quantitative Easing

The previous section has established the mechanism through which a quantitative easing shock affects term premia and economic activity in the Periphery and the Core. It has also illustrated that the magnitude of the effects relies crucially on the allocation of long-term bond purchases across regions. Although in practice the ECB allocates its purchases according to the capital key, i.e. by weighing purchases by GDP and population, the natural question that arises is whether this allocation is optimal? This section sets out to investigate this.

We set up an optimal quantitative easing policy under discretion, where the union-wide central bank cannot commit to future policy plans. As in Harrison (2017) we seek a Markov perfect policy in which both the optimal path of quantitative easing, $B_{L,t}^{CB}$, as well as its distribution across regions, s, is a function only of the relevant state variables in the model. As we show, the optimal share of quantitative easing will now not only reflect the different size of each region n, but will also be a function of the parameters dictating imperfect substitutability across all assets, the extent of price rigidities, and other dimensions of heterogeneity across regions.

We assume that the union central bank conducts QE according to the following two (linearized) rules:

$$\hat{b}_{L,t}^{CB}\frac{q_L}{y} = \hat{x}_t\left(\frac{q_L b_L}{y} + \frac{q_L^* b_L^*}{y^*}\frac{1-n}{n}\right), \hat{b}_{L,t}^{*,CB}\frac{q_L^*}{y^*} = \hat{x}_t^*\left(\frac{q_L b_L}{y}\frac{n}{1-n} + \frac{q_L^* b_L^*}{y^*}\right) \tag{5.1}$$

where x_t^P and x_t^C is the fraction of peripheral and core long-term bonds purchased by the central bank, similar to Harrison (2017).

To derive the optimal capital key weight, we maximize the welfare criterion of the Central Bank subject to seven constraints. These are, the two aggregate IS equations, the two Phillips curves, the ZLB constraint and the two conditions on the upper and the lower bound on the capital key weights. In normal times, when the ZLB does not bind the Lagrange multipliers on the ZLB and on the two aggregate demand equations, $\lambda_{C,t}$ and $\lambda_{C_t^*}$, are zero. We need to also impose the constraint that $\bar{x} \leq \hat{x}_t + \hat{x}_t^* \leq \bar{x}$. It is trivial to show that when the ZLB constraint does not bind, the optimal weights x_t and x_t^* are zero. In fact, when the ZLB does not bind $\lambda_{C,t} = \lambda_{C^*,t} = 0$ and, given that the weights x_t and x_t^* are not state variables. The objective function of the central bank is defined as follows:

$$\begin{aligned} L_t &= E_t \sum_{s=1}^{\infty} \beta^s \hat{X}_{t+s} Q \hat{X}_{t+s}{}' \\ &= \hat{X}_t Q \hat{X}_t' + \beta L_{t+1} \end{aligned} \tag{5.2}$$

where $\hat{X}_t$ is a vector containing the target variables and Q is a symmetric matrix of the weights. The latter are nonlinear functions of the model deep parameters. The minimization problem of the central bank at the ZLB has the following structure:

$$
\begin{aligned}
&\min_{\hat{\pi}_{h,t},\hat{\pi}^*_{f,t},\hat{C}_t,\hat{C}^*_t,\hat{p}_{h,t},\hat{p}_{f,t},\hat{\psi}_t,\hat{x}_t,\hat{x}^*_t,\hat{R}^U_t} \hat{X}_t Q \hat{X}_t' + \beta L_{t+1} \\
&- \lambda_t^C \left(\widehat{c}_t - \frac{h}{1+h}\widehat{c}_{t-1} - \frac{1}{1+h}\widehat{c}_{t+1} + \frac{1-h}{1+h}\left(PR_t - E_t\widehat{\pi}_{t+1}\right)\right) \\
&- \lambda_t^{C^*} \left(\widehat{c}^*_t - \frac{h^*}{1+h^*}\widehat{c}^*_{t-1} - \frac{1}{1+h^*}\widehat{c}^*_{t+1} + \frac{1-h^*}{1+h^*}\left(PR^*_t - E_t\widehat{\pi}^*_{t+1}\right)\right) \qquad (5.3) \\
&- \lambda_t^{\pi} \left(\widehat{\pi}_{h,t} - \beta E_t \widehat{\pi}_{h,t+1} - \frac{(1-\omega)(1-\beta\omega)}{\omega}\left(\widehat{mc}_t\right)\right) \\
&- \lambda_t^{\pi^*} \left(\widehat{\pi}^*_{f,t} - \beta E_t \widehat{\pi}^*_{f,t+1} - \frac{(1-\omega^*)(1-\beta\omega^*)}{\omega^*}\left(\widehat{mc}^*_t\right)\right) \\
&- \lambda_t^{R^U} \left(\hat{R}^U_t - \beta^{-1}\left(\pi - \psi\right) + 1\right) \\
&- \lambda_t^{\bar{x}} \left(\hat{x}_t + \hat{x}^*_t - \bar{x}\right) \\
&- \lambda_t^{\bar{x}} \qquad (5.4)
\end{aligned}
$$

The FOCs can be seen in the Appendix.

5.1 The optimal capital key

Since the start of its asset purchase program, the ECB has been implementing its purchases of eligible government bonds according to the capital key. The latter refers to the share of government bonds of a specific country in total purchases by the ECB. It is determined by the contribution of each country to the total balance sheet of the ECB. In this section, we derive the optimal capital key conditional on total QE the latter being expressed as a share of GDP. This does not determine the optimal QE in each region, as derived in the previous section. Instead, it determines the optimal share of peripheral and core bonds purchased by the union central bank. In section 2.5, we modeled purchases of peripheral bonds by:

$$q_{L,t}B^{cb}_{L,t} = \tilde{s}\upsilon_t(q_{L,t}B_{L,t} + q^*_{L,t}B^*_{L,t}) \qquad (5.5)$$

Therefore we are interested in deriving the optimal $\tilde{s}$ that minimizes the welfare loss function conditional on a given level of total asset purchases as a share of GDP, υ_t. Minimizing thus with respect to s yields:

$$\tilde{s} = \frac{1}{\Upsilon \upsilon_t} \qquad (5.6)$$

where Υ is given by:

$$\Upsilon = \left(\frac{q_L b_L}{y} + \frac{q_L^* b_L^*}{y^*}\frac{1-n}{n}\right)\left(\frac{y}{q_L}\right)\left[\frac{(1-\zeta_\alpha)\zeta_L}{b_{HL}} + \frac{(1-\zeta_\alpha^*)(1-\zeta_L^*)}{b_{HL}^*} - \left(\frac{b_S}{b_L} + q_L\right)\left(\frac{\zeta_\alpha \zeta_S}{b_{HS}} + \frac{\zeta_\alpha^*(1-\zeta_S^*)}{b_{HS}^*}\right)\right]$$
$$\times\left(-\left(n\xi\alpha_\alpha + (1-n)\,\xi^*\alpha_\alpha^*\right) + \beta\frac{(\psi\alpha)^2}{y^*+y}\left(\frac{nh\xi c}{y^2} + \frac{(1-n)\,h^*\xi^* c^*}{y^{*2}}\right)\right) \tag{5.7}$$

where the partial derivatives $\frac{\partial PR_t}{\partial \hat{x}_t}$ and $\frac{\partial PR_t^*}{\partial \hat{x}_t}$ as well as the parameters ξ, ξ^*, α_α and α_α^* are defined in detail in the appendix.

The derivation of the optimal capital key allows us to investigate how it is affected by changes in portfolio preferences, captured by parameters ζ_α and ζ_L, as well as by the elasticities of substitution between maturities and region of issuance, κ_α and κ_L.

Corollary 1. *The optimal capital key $\tilde{s}$ is decreasing in the share of the short-term portfolio, ζ_α, in the aggregate portfolio, α_t*

Proof: Appendix

This is consistent with the aim of a QE program, which becomes more effective when long-term debt supply is higher in the market. In contrast, when the long-term debt market is shallow, the program becomes less effective. A higher ζ_a implies a smaller long-term market for Peri; therefore dictates a smaller purchase for those bond by the central bank.

Corollary 2. *The optimal capital key $\tilde{s}$ is is decreasing in the elasticity of substitution between short- and long-term assets, κ_a.*

Proof: Appendix

This is a straightforward result since the closer substitutes the two asset classes are, the easier the households rebalance their portfolios and the faster they induce a fall in the term premium (see Figure 5.1).

Corollary 3. *The optimal capital key $\tilde{s}$ is decreasing in the elasticities of substitution between short- and long-term bonds, κ_a and κ_a^**

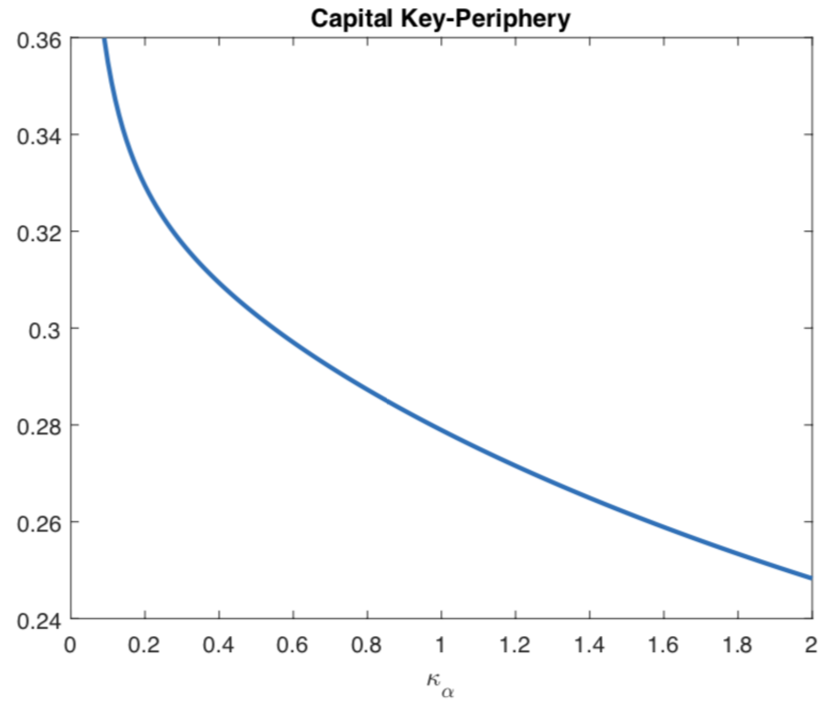
Proof: Appendix

Corollary 4. *The optimal capital key $\tilde{s}$ is decreasing in the home (periphery) elasticity of substitution between home (periphery) and foreign (core) long-term bonds, κ_L, and increasing in the foreign (core) elasticity of substitution between home (periphery) and foreign (core) long-term bonds, κ_L^*.*

Proof: Appendix

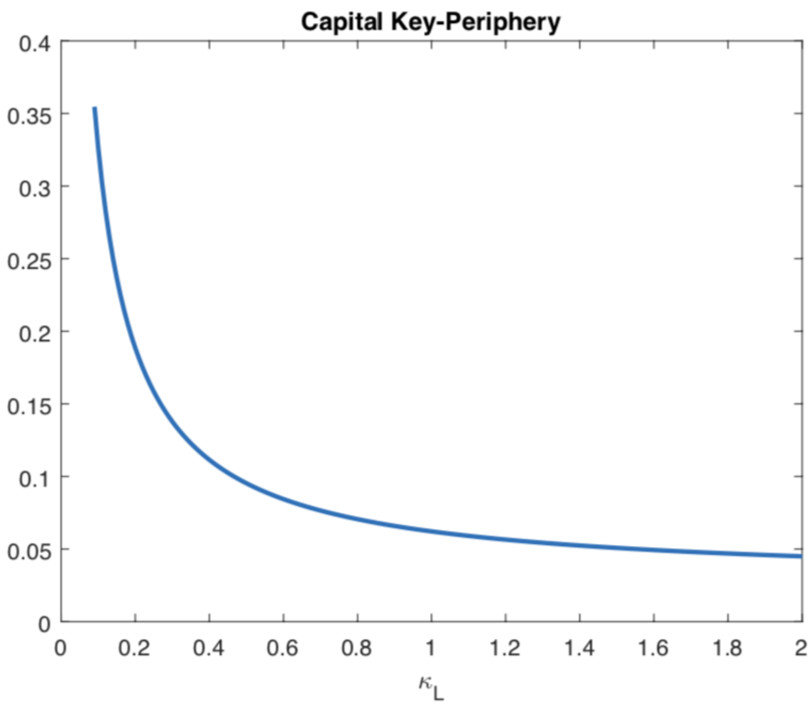
When Periphery resident can easily substitute their home bonds with foreign ones, purchasing Peri bonds will be less effective in reducing Peri term premium. Rather this would result in more spillovers to foreign (core) term premium, in which case central bank can increase the effectiveness by purchasing directly more Core bonds (see Figure 5.2).

Figure 5.1: Optimal capital key and κ_a



Notes: The Figure plots the optimal capital key, $\tilde{s}$, as a function of κ_a.

Figure 5.2: Optimal capital key and κ_L



Notes: The Figure plots the optimal capital key, $\tilde{s}$, as a function of κ_L.

6 Conclusion

This paper has provided a quantitative evaluation of the macroeconomic effects of quantitative easing in a currency union, using a two-region dynamic stochastic general equilibrium (DSGE) model consisting of the EA Periphery and the Core. In order to capture the effect of QE policies we have introduced a specific financial friction that limits investors' ability to arbitrage assets of different origin and maturity in their portfolio and isolates the *portfolio rebalancing* channel of QE. The asset portfolio is a CES aggregate of sub-portfolios of short-term and long term bonds. The sub-portfolios are in turn nested CES aggregates of domestic and foreign bonds of the same maturity.

In our model, central bank long-term asset purchases have effects on the real economy through the extent to which private investors are induced to re-establish the portfolio mix of short-term and long-term assets holdings, issued both domestically and abroad. These purchases affect asset prices and by extension real variables. First, purchases directly lower the term premium component of long-term yields. Second, term premiums spill over within the union through portfolio rebalancing of assets across borders. And third, lower outstanding government debt held by private agents also lowers the risk premium on these assets. Ultimately, reduced savings strengthen contemporaneous consumption demand.

Using data from the ECB's Security Holdings Statistics, we have calibrated the portfolio elasticities so that our quantitative analysis replicates four targets from the data: i)-ii) stable long-term bond shares over the QE period both for the Periphery and Core, iii) similar changes in the share of short-term bonds over the QE period (0.48 to 0.57 for the Periphery, and from 0.67 to 0.81 in the Core) from 2015-2018, and iv) a 65 basis point decline in the union wide term premium. The values suggest that financial frictions are higher in the Periphery so for a QE shock of given size across regions Periphery portfolios rebalance to a larger degree than in Core. In our model a 10% QE shock generates a larger drop in the term premium in the Periphery (75 basis points (bp)) than in the Core (30bp), implying a more expansionary effect on Periphery consumption than in that of the Core.

After having established that the macroeconomic effects rely crucially on the way long-term bond purchases are allocated across regions, we set up an optimal quantitative easing policy under discretion, where the union-wide central bank cannot commit to future policy plans.We have obtained a Markov perfect policy in which both the optimal path of quantitative easing, as well as its distribution across regions, is a function only of the relevant state variables in the model. As we have shown, the optimal share of quantitative easing does not only reflect the different size of each region, but is also a function of the parameters dictating imperfect substitutability across all assets, the extent of price rigidities, and other dimensions of heterogeneity across regions.

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A Optimal Quantitative Easing

The FOCs from 5.3 write as follows:

$$\hat{\pi}_{h,t}: \; 0 = n\xi_{\pi}\hat{\pi}_{h,t} - \lambda_t^{\pi}$$

$$\hat{\pi}^*_{f,t}: \; 0 = (1-n)\,\xi^*_{\pi^*}\hat{\pi}^*_{f,t} - \lambda_t^{\pi*}$$

$$\hat{c}_t : 0 = \left(n\xi\alpha_c + (1-n)\,\xi^*\alpha_c^* + \beta\left(\frac{nh\xi c}{y^2} + \frac{(1-n)\,h^*\xi^*c^*}{y^{*2}}\right)\frac{c^2}{y(y^*+y)} + \frac{(1-n)\,y\xi^*}{(y^*+y)}\left(\frac{c_h}{y}\right)^2 + \frac{ny^*\xi}{(y^*+y)}\left(1-\frac{c_f^*+g^*}{y^*}\right)^2\right)\hat{c}_t$$
$$-\lambda_t^c + \frac{(1-\omega)(1-\omega\beta)}{\omega(1-h)}\lambda_t^{\pi} + \alpha_{cp_h}\hat{p}_{h,t} + \alpha^*_{cp_f}\hat{p}_{f,t} + (n\xi_{\alpha_{cc^*}} + (1-n)\xi^*\alpha^*_{cc^*})\,\hat{c}^*_t$$

$$\hat{c}^*_t : 0 = \left(n\xi\alpha_{c^*} + (1-n)\,\xi^*\alpha^*_{c^*} + \beta\left(\frac{nh\xi c}{y^2} + \frac{(1-n)\,h^*\xi^*c^*}{y^{*2}}\right)\frac{c^{*2}}{y(y^*+y)} + \frac{ny^*\xi}{(y^*+y)}\left(\frac{c_f^*}{y^*}\right)^2 + \frac{(1-n)\,y\xi^*}{(y^*+y)}\left(1-\frac{c_h+g}{y}\right)^2\right)\hat{c}^*_t$$
$$-\lambda_t^{c^*} + \frac{(1-\omega^*)(1-\omega^*\beta)}{\omega^*(1-h^*)}\lambda_t^{\pi^*} + \alpha^*_{c^*p_f}\hat{p}_{f,t} + \alpha_{c^*p_h}\hat{p}_{h,t} + (n\xi_{\alpha_{cc^*}} + (1-n)\xi^*\alpha^*_{cc^*})\,\hat{c}_t$$

$$\hat{p}_{h,t}: \; 0 = \left(n\xi\alpha_{p_h} + \frac{\xi^*y(1-n)}{y^*(y^*+y)}\left(\frac{c_h}{y}(\kappa-\kappa^*) + \kappa^*(1-\frac{g}{y})\right)^2\right)\hat{p}_{h,t} + \alpha_{cp_h}\hat{c}_t + \alpha_{c^*p_h}\hat{c}^*_t + \frac{(1-\omega)(1-\omega\beta)}{\omega(1-h)}\lambda_t^{\pi}$$

$$\hat{p}_{f,t}: \; 0 = \left((1-n)\xi\alpha^*_{p_f} + \frac{\xi y^*n}{y(y^*+y)}\left(\frac{c_f^*}{y^*}(\kappa^*-\kappa) + \kappa(1-\frac{g^*}{y^*})\right)^2\right)\hat{p}_{f,t} + \alpha^*_{cp_f}\hat{c}_t + \alpha^*_{c^*p_f}\hat{c}^*_t + \frac{(1-\omega^*)(1-\omega^*\beta)}{\omega^*(1-h^*)}\lambda_t^{\pi*}$$

$$\hat{\psi}_t: \; 0 = \left(n\xi\alpha_{\psi} + (1-n)\,\xi^*\alpha^*_{\psi} + \beta\left(\frac{nh\xi c}{y^2} + \frac{(1-n)\,h^*\xi^*c^*}{y^{*2}}\right)\frac{1-c^*-(\Psi-\psi^*\alpha^*)}{y(y^*+y)}\right)\hat{\psi}_t - \frac{(1-\omega^*)(1-\omega^*\beta)}{\omega^*}\lambda_t^{\pi^*}$$

$$\hat{x}_t: \; 0 = (n\xi\alpha_{\alpha} + (1-n)\,\xi^*\alpha^*_{\alpha})\left(\frac{q_Lb_L}{y} + \frac{q_L^*b_L^*}{y^*}\frac{1-n}{n}\right)\left(\frac{\zeta_L y}{q_Lb_{HL}} + \frac{(1-\zeta_L)y}{q_Lb^*_{HL}}\right)\left(1+\beta\frac{(\psi\alpha)^2}{y^*+y}\left(\frac{nh\xi c}{y^2} + \frac{(1-n)\,h^*\xi^*c^*}{y^{*2}}\right)\right)\hat{x}_t$$
$$-\frac{1-h}{1+h}\frac{\partial PR_t}{\partial \hat{x}_t}\lambda_t^c - \frac{1-h^*}{1+h^*}\frac{\partial PR_t^*}{\partial \hat{x}_t}\lambda_t^{c^*} - \lambda_t^{\bar{x}} - \lambda_t^{\bar{x}}$$

where the partial derivatives $\frac{\partial PR_t}{\partial \hat{x}_t}$, $\frac{\partial PR^*_t}{\partial \hat{x}_t}$, $\frac{\partial PR_t}{\partial \hat{x}^*_t}$ and $\frac{\partial PR^*_t}{\partial \hat{x}^*_t}$ are specified as follows:

$$\begin{aligned}
\frac{\partial PR_t}{\partial \hat{x}_t} &= \frac{\partial PR_t}{\partial \hat{T}_t}\frac{\partial \hat{T}_t}{\partial \hat{x}_t} + \frac{\partial PR_t}{\partial \hat{T}^*_t}\frac{\partial \hat{T}^*_t}{\partial \hat{x}_t} \\
&= (1-\zeta_\alpha)\zeta_L\frac{\partial \hat{T}_t}{\partial \hat{x}_t} + (1-\zeta_\alpha)(1-\zeta_L)\frac{\partial \hat{T}^*_t}{\partial \hat{x}_t} \\
&= -(1-\zeta_\alpha)\zeta_L\left[\left(\frac{q_Lb_L}{y} + \frac{q_L^*b_L^*}{y^*}\frac{1-n}{n}\right)\left(\left(\frac{1}{\kappa_\alpha} - \frac{1}{\kappa_L}\right)\left(\frac{\zeta_L y}{q_Lb_{HL}}\right) + \frac{1}{\kappa_L}\frac{1}{b_{HL}}\right)\right] \\
&\quad - (1-\zeta_\alpha)(1-\zeta_L)\left(\frac{q_Lb_L}{y} + \frac{q_L^*b_L^*}{y^*}\frac{1-n}{n}\right)\left(\frac{1}{\kappa^*_\alpha} - \frac{1}{\kappa^*_L}\right)\frac{(1-\zeta^*_L)y}{q_Lb^*_{HL}} \\
&= -(1-\zeta_\alpha)\left(\frac{q_Lb_L}{y} + \frac{q_L^*b_L^*}{y^*}\frac{1-n}{n}\right)\left[\zeta_L\left(\left(\frac{1}{\kappa_\alpha} - \frac{1}{\kappa_L}\right)\left(\frac{\zeta_L y}{q_Lb_{HL}}\right) + \frac{1}{\kappa_L}\frac{1}{b_{HL}}\right) + (1-\zeta_L)\left(\frac{1}{\kappa^*_\alpha} - \frac{1}{\kappa^*_L}\right)\frac{(1-\zeta^*_L)y}{q_Lb^*_{HL}}\right]
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\frac{\partial PR^*_t}{\partial \hat{x}_t} &= \frac{\partial PR^*_t}{\partial \hat{T}_t}\frac{\partial \hat{T}_t}{\partial \hat{x}_t} + \frac{\partial PR^*_t}{\partial \hat{T}^*_t}\frac{\partial \hat{T}^*_t}{\partial \hat{x}_t} \\
&= (1-\zeta^*_\alpha)(1-\zeta^*_L)\frac{\partial \hat{T}_t}{\partial \hat{x}_t} + (1-\zeta^*_\alpha)\zeta^*_L\frac{\partial \hat{T}^*_t}{\partial \hat{x}_t} \\
&= -(1-\zeta^*_\alpha)\left(\frac{q_Lb_L}{y} + \frac{q_L^*b_L^*}{y^*}\frac{1-n}{n}\right)\left[(1-\zeta^*_L)\left(\left(\frac{1}{\kappa_\alpha} - \frac{1}{\kappa_L}\right)\left(\frac{\zeta_L y}{q_Lb^*_{HL}}\right) + \frac{1}{\kappa_L}\frac{1}{b_{HL}}\right) + \zeta^*_L\left(\frac{1}{\kappa^*_\alpha} - \frac{1}{\kappa^*_L}\right)\frac{(1-\zeta^*_L)y}{q_Lb^*_{HL}}\right]
\end{aligned} \tag{A.2}$$

$$
\begin{aligned}
\frac{\partial PR_t}{\partial \hat{x}_t^*} &= \frac{\partial PR_t}{\partial \hat{T}_t}\frac{\partial \hat{T}_t}{\partial \hat{x}_t^*} + \frac{\partial PR_t}{\partial \hat{T}_t^*}\frac{\partial \hat{T}_t^*}{\partial \hat{x}_t^*} \\
&= (1-\zeta_\alpha)\zeta_L \frac{\partial \hat{T}_t}{\partial \hat{x}_t^*} + (1-\zeta_\alpha)(1-\zeta_L)\frac{\partial \hat{T}_t^*}{\partial \hat{x}_t^*} \\
&= -(1-\zeta_\alpha)\zeta_L\left(\frac{q_L b_L}{y}\frac{n}{1-n} + \frac{q_L^* b_L^*}{y^*}\right)\left(\frac{1}{\kappa_\alpha} - \frac{1}{\kappa_L}\right)\left(\frac{(1-\zeta_L)y^*}{q_L^* b_{FL}}\right) \\
&\quad - (1-\zeta_\alpha)(1-\zeta_L)\left[\left(\frac{q_L b_L}{y}\frac{n}{1-n} + \frac{q_L^* b_L^*}{y^*}\right)\left(\left(\frac{1}{\kappa_\alpha^*} - \frac{1}{\kappa_L^*}\right)\frac{\zeta_L^* y^*}{q_L^* b_{FL}^*} + \frac{1}{\kappa_L^*}\frac{1}{b_{FL}^*}\right)\right] \\
&= -(1-\zeta_\alpha)\left(\frac{q_L b_L}{y}\frac{n}{1-n} + \frac{q_L^* b_L^*}{y^*}\right)\left[(1-\zeta_L)\left(\left(\frac{1}{\kappa_\alpha^*} - \frac{1}{\kappa_L^*}\right)\frac{\zeta_L^* y^*}{q_L^* b_{FL}^*} + \frac{1}{\kappa_L^*}\frac{1}{b_{FL}^*}\right) + \zeta_L\left(\frac{1}{\kappa_\alpha} - \frac{1}{\kappa_L}\right)\frac{(1-\zeta_L)y^*}{q_L^* b_{FL}}\right]
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\frac{\partial PR_t^*}{\partial \hat{x}_t^*} &= \frac{\partial PR_t^*}{\partial \hat{T}_t}\frac{\partial \hat{T}_t}{\partial \hat{x}_t^*} + \frac{\partial PR_t^*}{\partial \hat{T}_t^*}\frac{\partial \hat{T}_t^*}{\partial \hat{x}_t^*} \\
&= (1-\zeta_\alpha^*)(1-\zeta_L^*)\frac{\partial \hat{T}_t}{\partial \hat{x}_t^*} + (1-\zeta_\alpha^*)\zeta_L^*\frac{\partial \hat{T}_t^*}{\partial \hat{x}_t^*} \\
&= -(1-\zeta_\alpha^*)\left(\frac{q_L b_L}{y}\frac{n}{1-n} + \frac{q_L^* b_L^*}{y^*}\right)\left[\zeta_L^*\left(\left(\frac{1}{\kappa_\alpha^*} - \frac{1}{\kappa_L^*}\right)\frac{\zeta_L^* y^*}{q_L^* b_{FL}^*} + \frac{1}{\kappa_L^*}\frac{1}{b_{FL}^*}\right) + (1-\zeta_L^*)\left(\frac{1}{\kappa_\alpha} - \frac{1}{\kappa_L}\right)\frac{(1-\zeta_L)y^*}{q_L^* b_{FL}}\right]
\end{aligned}
\tag{A.4}
$$

Finally, the derivatives $\frac{\partial PR_t}{\partial \hat{R}_t^U}$ and $\frac{\partial PR_t^*}{\partial \hat{R}_t^U}$ are specified as follows:

$$\frac{\partial PR_t}{\partial \hat{R}_t^U} = \zeta_\alpha\left(1-\zeta_S\right) + \left(1-\zeta_\alpha\right)\left(1-\zeta_L\right) \tag{A.5}$$

$$\frac{\partial PR_t^*}{\partial \hat{R}_t^U} = \zeta_\alpha^*\zeta_S^* + \left(1-\zeta_\alpha^*\right)\zeta_L^* \tag{A.6}$$

The parameters in the equations above are specified as follows:

$$\xi = \frac{c}{y}\frac{(1-h)(1-\omega)(1-\omega\beta)}{\omega\theta},\ \xi^* = \frac{c^*}{y^*}\frac{(1-h^*)(1-\omega^*)(1-\omega^*\beta)}{\omega^*\theta^*}$$

$$\alpha_\alpha = \frac{C(\psi\alpha)^2}{(Y^*+Y)Y^2},\ \alpha_\alpha^* = \frac{C^*(\psi\alpha)^2}{(Y^*+Y)Y^{*2}}$$

$$\alpha_{\alpha^*} = \frac{C(\psi^*\alpha^*)^2}{(Y^*+Y)Y^2},\ \alpha_{\alpha^*}^* = \frac{C^*(\psi^*\alpha^*)^2}{(Y^*+Y)Y^{*2}}$$

$$\alpha_c = \left(\frac{c_h}{y}\right)^2\left(\frac{c}{y^*+y}+1\right) - \frac{c^2}{y(y^*+y)},\ \alpha_c^* = \left(1-\frac{c_f^*+g^*}{y^*}\right)^2\left(\frac{c^*}{y^*+y}+1\right) - \frac{c^2}{y^*(y^*+y)}$$

$$\alpha_{c^*} = \left(1-\frac{c_h+g}{y}\right)^2\left(\frac{c}{y^*+y}+1\right) - \frac{c^{*2}}{y(y^*+y)},\ \alpha_{c^*}^* = \left(\frac{c_f^*}{y^*}\right)^2\left(\frac{c^*}{y^*+y}+1\right) - \frac{c^{*2}}{y^*(y^*+y)}$$

$$\alpha_{p_h} = \left(\frac{c_h}{y}(\kappa-\kappa^*) + \kappa^*\left(1-\frac{g}{y}\right)\right)^2\left(\frac{c}{y^*+y}+1\right) - \frac{(gp_h)^2 c}{y(y^*+y)}$$

$$\alpha_{p_f}^* = \left(\frac{c_f^*}{y^*}(\kappa^*-\kappa) + \kappa\left(1-\frac{g^*}{y^*}\right)\right)^2\left(\frac{c^*}{y^*+y}+1\right) - \frac{(g^*p_f)^2 c^*}{y^*(y^*+y)}$$

$$\alpha_{cp_h} = -2\left\{\frac{n\xi c_h}{y}\left[\frac{c_h}{y}(\kappa-\kappa^*) + \kappa^*\left(1-\frac{g}{y}\right)\right]\left(\frac{c}{y^*+y}+1\right) + \frac{(1-n)\xi^* y}{y^*+y}\frac{c_h}{y}\left(\frac{c_h}{y}(\kappa-\kappa^*) + \kappa^*(1-\frac{g}{y})\right)\right\}$$

$$\alpha_{c^*p_h} = -2\{n\xi\left(1-\frac{c_h+g}{y}\right)\left[\frac{c_h}{y}(\kappa-\kappa^*)+\kappa^*\left(1-\frac{g}{y}\right)\right]\left(\frac{c}{y^*+y}+1\right)$$
$$+\frac{(1-n)\xi^*y}{y^*+y}\left(1-\frac{c_h+g}{y}\right)\left[\frac{c_h}{y}(\kappa-\kappa^*)+\kappa^*\left(1-\frac{g}{y}\right)\right]\}$$

$$\alpha^*_{cp_f} = -2\{(1-n)\xi^*\left(1-\frac{c_f^*+g^*}{y^*}\right)\left[\frac{c_f^*}{y^*}(\kappa^*-\kappa)+\kappa\left(1-\frac{g^*}{y^*}\right)\right]\left(\frac{c^*}{y^*+y}+1\right)$$
$$+\frac{n\xi y^*}{y^*+y}\left(1-\frac{c_f^*+g^*}{y^*}\right)\left[\frac{c_f^*}{y^*}(\kappa^*-\kappa)+\kappa\left(1-\frac{g^*}{y^*}\right)\right]\}$$

$$\alpha^*_{c^*p_f} = -2\left\{(1-n)\xi^*\frac{c_f^*}{y^*}\left[\frac{c_f^*}{y^*}(\kappa^*-\kappa)+\kappa\left(1-\frac{g^*}{y^*}\right)\right]\left(\frac{c^*}{y^*+y}+1\right)+\frac{n\xi y^*}{y^*+y}\frac{c_f^*}{y^*}\left[\frac{c_f^*}{y^*}(\kappa^*-\kappa)+\kappa\left(1-\frac{g^*}{y^*}\right)\right]\right\}$$

$$\alpha_{cc^*} = 2\left\{\frac{c_h}{y}\left(1-\frac{c_h+g}{y}\right)\left(\frac{c}{y^*+y}+1\right)\right\}$$

$$\alpha^*_{cc^*} = 2\left\{\frac{c_f^*}{y^*}\left(1-\frac{c_f^*+g^*}{y^*}\right)\left(\frac{c^*}{y^*+y}+1\right)\right\}$$

$$\alpha_\psi = \left(\frac{c}{y(y^*+y)}\right)\left(y^*-\frac{c^*}{y}-\Psi+\psi\alpha\right),\ \alpha^*_\psi = \left(\frac{c^*}{y^*(y^*+y)}\right)\left(y-\frac{c}{y^*}-\Psi^*+\psi^*\alpha^*\right)$$